remains a skeptic here. I believe that the observational data are too feeble, and the physical theory too tentative to support strong and conclusive beliefs.

"It seems to be a good principle that the prediction of a singularity by a physical theory indicates that the theory has broken down, i.e. it no longer provides a correct description of observations. The question is: when does General Relativity break down? One would expect it to break down anyway when quantum gravitational effects become important. . . . This would correspond to a density of 10^{94} gm cm^{-3}. However one might question whether a Lorentz manifold is an appropriate model for space-time on length scales of this order."

Cosmology is not a "hard experimental science" in the sense of aerodynamics, or even macro-economics. That is, the experimenter does not have access to an effective input control to the physical system. While there are plenty of observed data from millions of stars, they are not necessarily the data one might want if there were a choice. Unless we greatly modify our philosophy of scientific knowledge, cosmology must remain a speculation.

I recall once hearing a lecture on economics where the authority asserted something like, "since the coefficient of σ^2 is negative, we must expect the government to raise taxes", and I said to myself, "Hey, wait just a minute–let's multiply by -1". I had somewhat the same emotion when I read that the coefficient of σ^2 in Raychaudhuri's equation is negative, therefore we must all fall eventually into a black hole and this is our final fate. But even if this is the case, it might not be too dismal. Remember, Alice found a Wonderland.

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Topics in stochastic processes, by Robert B. Ash and Melvin F. Gardner, Probability and Mathematical Statistics, vol. 27, Academic Press, New York, 1975, viii + 321 pp., $34.50.

The theory of stochastic processes has mushroomed in the last twenty years; not only because of its intrinsic interest, but also because it is closely connected with so many different areas of mathematics. It feeds on analytic techniques from measure theory, Fourier analysis, semigroups of operators and spectral theory, potential theory, ergodic theory; and in turn it has applications to topology, functional inequalities, differential equations, information theory and prediction theory, and through the stochastic integral to several areas of mathematical physics. Thus stochastic processes is a good modern example of an area of mathematics which has been stimulated by its applications, while itself leading to extensive research in more established areas in order to develop the techniques needed.

The essential apparatus of mathematical building bricks is both extensive and deep. There is therefore no hope of writing a self contained text book of acceptable length which covers more than a small subset of the theory. The subset chosen by Ash and Gardner is made by selecting some special processes with interesting applications to physics or engineering and developing the framework needed for studying these processes. They start the investigation of several types of stochastic process without getting very far in any direction. The book assumes basic measure theory and the standard results about absolute integrals as well as basic probability theory. The authors also use a variety of results from complex analysis and Fourier analysis, but these are summarised conveniently in two Appendices.

The start of our intuitive understanding of stochastic processes is largely the result of the work of Paul Lévy [**3**], who seemed to think of a process from the point of view of a particle which at time t_0 was sitting at $X(t_0)$ and then continued to move along $X(t)$ as t grew. Lévy's intuition was largely based on a deep study of mathematical Brownian motion. The modern theory of continuous parameter Markov processes has grown out of Lévy's work. Intuitively, we should think of a Markov process as one in which the future $X(t)$, $t > t_0$, is independent of the past $X(t)$, $t < t_0$, given the present $X(t_0)$. The so-called 'strong Markov property' extends this definition to random times t_0 which are determined by the process for values of $t \leqslant t_0$. Lévy used the strong Markov property as well as the Markov property without formulating them precisely. A rigorous theory of Markov processes requires careful measure theoretic formulation–and this was first put together systematically by J. L. Doob [**1**] in 1953. In order to make simple properties of the sample path, such as $\{a \leqslant X(t) \leqslant b$ for all t in an interval $I\}$ measurable, one has to assume separability–which can be thought of as a condition that $X(t)$ is completely determined by values of t in a countable set. This leads on to the establishment of measurability properties for $X(t, \omega)$ in the product space $T \times \Omega$–which are needed to justify the use of integration methods. Ash and Gardner, in Chapter 4, obtain these deep properties for Markov processes with independent increments. They use the martingale convergence theorem as a tool to establish sample path properties, and look in more detail at the case of Brownian motion which is historically so important, obtaining results like the law of iterated logarithm which describes the asymptotic growth rate of the path.

There is an important class of stochastic processes which can be studied rigorously without all the apparatus of separability, measurability and continuity discussed in Chapter 4–this is the class of complex-valued processes $X(t)$ for which $|X(t)|^2$ is always integrable. These are called L^2-processes. We can then define

$$K(s, t) = \mathrm{Cov}\left[X(s), X(t)\right] = E\left[(X(s) - m(s))\left(\overline{X(t) - m(t)}\right)\right]$$

where $m(t) = E(X(t))$ exists, and call it the covariance function of the process. In the main Ash and Gardner restrict their discussion to the case of (wide sense) stationary processes for which

$$m(t) \equiv c \quad \text{and} \quad K(t, s) = K(t + h, s + h).$$

In this case $K(s, t)$ is determined by $K(t) = K(s + t, s)$. The celebrated theorem of Bochner tells us that a complex valued K on $\mathbf{R}$ that is continuous at the origin is the covariance of a stationary L^2-process if and only if there is a finite Borel measure μ on $\mathbf{R}$ (called the spectral measure) such that

$K(t) = \int_R e^{ity} \, d\mu(y)$. The L^2-process is not determined by K, and it is possible to choose one that is Gaussian, that is, such that $X(t_1)$, $X(t_2), \ldots, X(t_n)$ have a jointly Gaussian distribution. The expansion of an L^2-process $X(t)$ on a finite interval $a \leqslant t \leqslant b$ can be given in terms of the eigenfunctions of $K(s, t)$ in $L^2[a, b]$, and this ties in the theory of L^2-processes to that of Hilbert spaces. In particular the usual spectral theory can be applied to the spectrum μ when the process is stationary. Ash and Gardner give an illuminating account of this theory in Chapters 1 and 2. They use the theory to attack the prediction problem–which we can think of as seeking to predict $X(t_0 + s)$ given the process $X(t)$ for $t \leqslant t_0$. They develop their methods in a number of cases which have application to engineering. A fuller treatment of the prediction problem is given in Gihman and Skorohod **[2]**.

We can think of ergodic theory as the study of measure preserving transformations T on a measure space $(\Omega, \mathfrak{F}, \mu)$ in which the only T-invariant sets are those with zero or full measure. When $\mu(\Omega) = 1$ the 'strong law of large numbers' states that

$$\frac{1}{n} \sum_{i=0}^{n-1} X_i \to E(X)$$

almost surely as $x \to \infty$, where X is a random variable with finite expectation and X_i are independent realisations of X. In Chapter 3, Ash and Gardner give a standard development of ergodic theory as far as a proof of the pointwise ergodic theorem which, on a probability space can be formulated as

$$\frac{1}{n} \sum_{i=0}^{n-1} X(T^i \omega) \to E(X)$$

almost surely, for any random variable X with finite expectation. Thus the pointwise ergodic theorem is very closely related to the strong law of large numbers. The chapter then discusses briefly the application of ergodic theory to Markov chains and goes on to obtain the Shannon-McMillan theorem on the convergence of

$$-\frac{1}{n} \log p(X_0, \ldots, X_{n-1})$$

in the case where the X_i are the discrete coordinate random variables of a two sided shift T. When T is ergodic, this limit is a constant H which can be identified as the entropy (or degree of uncertainty) given by

$$H(X) = -\sum_x p(x) \log p(x),$$

where $p(x) = P\{X = x\}$. This is a basic result in the modern theory of information.

The final chapter of the book is a brief introduction to the study of differential equations with random coefficients. For this purpose the Itô stochastic integral is established, as this is the basic tool needed. Think of one dimensional Brownian motion $B(t, \omega)$ as defined on a probability space $(\Omega, \mathfrak{F}, P)$ and $f : T \times \Omega \to \mathbf{R}$ as a function with values $f(t,\omega)$. The object is to define $\int_a^b f(t, \omega) \, dB(t, \omega)$ under appropriate conditions. This cannot be defined as a Lebesgue-Stieltjes integral since $B(t, \omega)$ is not of bounded variation. Itô's definition is obtained as a limit of step function approximations which works when (i) f is $\mathfrak{B} \times \mathfrak{F}$ measurable; (ii) for each $t \in [a, b]$, $f(t) \in L^2(\Omega, P)$ and $\int_a^b E(|f(t)|^2)\, dt < \infty$; (iii) for each $t \in [a, b]$, $f(t, \cdot)$ is measurable $\mathfrak{F}(t)$, the σ-field generated by $\{B(s), s \leq t\}$. Note that condition (ii) restricts the average size of $|f(t)|$ and (iii) says that the dependence of $f(t, \omega)$ on ω is restricted to information about the past and present values of $B(s, \omega)$. This chapter does no more than give a taste of a large subject with important applications. An interested reader would go on to consult the book by McKean [**4**].

The reviewer enjoyed his commision to read the book. He suspects that the book will have limited value as a reference work because no topic is pushed very far. It does have a good selection of examples worked out in the text as well as problems at the end of each chapter, which are provided with outline solutions. This means that a competent graduate student or an analyst unfamiliar with stochastic processes would profit greatly by careful study of the book. It would make a good text for an advanced graduate course provided the lecturer was satisfied with the topics selected. The authors have provided a valuable new perspective on a variety of important analytic tools used for the study of stochastic processes.

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Homotopy theory; an introduction to algebraic topology, by Brayton Gray, Academic Press, New York, 1976, xiii + 368 pp., $22.00.

"This book is an exposition of elementary algebraic topology from the point of view of a homotopy theorist." It is with this sentence that the Preface to Brayton Gray's book begins, so perhaps we would be well advised to learn something of the homotopy theorist's point of view before examining the contents of the book itself.

In a vague sense homotopy theory studies properties of topological spaces that remain invariant under a continuous deformation. The achievements of the theory stem from the fact that so many seemingly rigid problems are really homotopy theoretic in nature.

Around the turn of the century, during the formative period of algebraic topology, Poincaré introduced [**9**] (among other things) the homology (groups) of a polyhedron. A polyhedron is a configuration of basic convex sets called simplexes, and it was from the combinatorial properties of the configuration that the homology of a polyhedron was defined. It then became essential to demonstrate that these combinatorially defined invariants were in