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BANACH LATTICES WITH A WEAK ORDER UNIT

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Starting from a result due to H. Rosenthal (see Lemma 1.3 in [**6**]), we present a new result which outlines a geometrical condition for the existence of a weak order unit (i.e., a total element) in the dual of a Banach lattice: the nonexistence of a lattice isomorph of a space $l_1(\Gamma)$ for Γ an uncountable set (see Theorem 2 below). As a consequence we obtain that the dual of a Banach space E having local unconditional structure (as defined in [1] has the Radon-Nikodým property iff E does not contain a copy of l_1. R. C. James constructed, in [3], an example of separable Banach space J with a nonseparable dual such that c_0 and l_1 do not embed in J. Consequently J does not have local unconditional structure.

Let L be the class of all Banach spaces X satisfying the following two conditions:

($\mathsf{L}1$) X' is an order complete Banach lattice;

($\mathsf{L}2$) there is a $v \in X''$ such that $x \in X$, $|x| \wedge |v| = 0$ implies $x = 0$.

Each Banach lattice with a weak order unit, or each predual of an $L_1(\mu)$ space, belongs to L.

1. LEMMA. *If $X \in \mathsf{L}$, then there exists an order complete Banach lattice E with a weak order unit and a lattice isometry i: $X' \longrightarrow E'$ such that*: (a) $i(X')$ *is complemented in E', and* (b) $i(X')$ *is formed by order continuous functionals on E.*

Hint. Consider for E the band generated by v in X''.

If Z is an order complete Banach lattice and $A \subset Z$ is a closed subspace, then we shall denoted by $\Sigma(A)$ an order complete closed sublattice of the band generated by A such that $A \subset \Sigma(A)$.

2. THEOREM. *Let $E \in \mathsf{L}$ and let $A \subset E'$ be a closed subspace. Then either*:

(i) *$\Sigma(A)$ has a weak order unit $u' > 0$; or,*

(ii) *A contains an isomorph of $l_1(\Gamma)$ (for Γ an uncountable set) that is complemented in $\Sigma(A)$, and $\Sigma(A)$ contains a lattice isomorph of $l_1(\Gamma)$.*

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Note. The hypothesis that $E \in L$ is crucial; e.g., consider the case $E = l_2(2^N)$.

3. COROLLARY. *If E is a Banach lattice with a weak order unit, then either E' has a weak order unit or E' contains a lattice isomorph of a non-separable $l_1(\Gamma)$ space.*

4. REMARK. It is possible that E and E' both have a weak order unit and E' contains a complemented isomorph (but not a lattice isomorph) of $l_1(2^N)$. For example, consider $E = (\Sigma l_\infty(n))_{l_1}$. (See [2] for details.)

5. REMARK. If E is a Banach lattice, then condition (i) can be restated in terms of absolute continuity:

(i′) For every $\epsilon > 0$, every $x \in E_+$, and every $a' \in \Sigma(A)$, there is a $\delta = \delta(\epsilon, x, a') > 0$ such that $|y| \leqslant x$, $u'(|y|) < \delta$, implies $|a'(y)| < \epsilon$.

6. REMARK. If $E \in L$ and E does not contain a complemented copy of l_1, then E' does not contain a copy of c_0 (see [**6**, Corollary 1.2]) and thus (see [7, Theorem 3.7]) each order interval of E' is weakly relatively compact. In this case (i) can be restated as follow:

(i″) A is contained in a weakly compactly generated (complemented) sublattice having a weak order unit.

As a consequence we retrieve Lemma 1.3 in [**6**].

Recall that a Banach space Z has **local unconditional structure** (l.u.st.) if there exists a positive $\lambda > 0$ and a directed net $\{Z_a\}_a$ of finite dimensional subspaces such that $Z = \bigcup Z_a$ and each Z_a has a basis whose unconditional constant is $\leqslant \lambda$. The usual Banach spaces all have l.u.st. We next extend a well-known result due to James for Banach spaces with an unconditional basis.

7. THEOREM. *For a separable Banach space E having l.u.st., the following assertions are equivalent*:

(a) *E' has the Radon-Nikodým property* (*i.e., every integral operator from a space $C(S)$ into E' is nuclear*);

(b) *E does not contain an isomorph of l_1*;

(c) *E' is weakly compactly generated.*

Therefore, if E is a separable Banach space with l.u.st., then E' is separable if and only if E contains no isomorph of l_1.

If Z has l.u.st. and Y is a separable subspace of Z, then there is a separable Banach space X with l.u.st. such that $Y \subset X \subset Z$. (See the proof of Lemmas 3.1 and 3.2 in [**4**].) Consequently, the equivalence (a) $\Longleftrightarrow$ (b) holds also in the nonseparable case.

The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are well known. We next prove that (b) $\Rightarrow$ (c). By virtue of Theorem 2.1 and Corollary 2.2 in [**1**], there exists a separable Banach lattice $L \supset E$ and an isomorphism ϕ from E' into L' such that $i' \circ \phi = 1_{E'}$, where i: $E \longrightarrow L$ denotes the canonical embedding. A result due to Pełczyński and Hagler (see Studia Math. **46** (1973), 35–42) implies that E' does not contain a copy of $l_1(\Gamma)$, for Γ an uncountable set, so by our Theorem 2 above, $\phi(E')$ is contained in the band generated by a positive $u' \in L'$. We conclude by using a technique due to H. P. Lotz. For each $x' \in L', x' \geqslant 0$, let us denote by $I_{x'}$, the ideal generated by x' and normed by $|||\cdot||| = \inf\{\lambda > 0; |\cdot| \leqslant \lambda x'\}$. Then $I_{x'}$, is lattice isometric to a $C(S)$ space. By Corollary 1.2 in [6], E' does not contain a copy of c_0 and thus the composition $j_{x'}: I_{x'} \longrightarrow L' \xrightarrow{i'} E'$ is weakly compact (see [7, Theorem 3.7]). Since $I_{x'}$ has the Dunford-Pettis property, $j_{x'}$ maps decreasing sequences of positive elements of $I_{x'}$ into convergent sequences of elements of E'. Since $j_{x'}$ is w'-continuous (in fact $j_{x'}$ is an adjoint), we obtain that $j_{x'}$ is order σ-continuous for each $x' \geqslant 0$. Then $E' = \overline{\operatorname{Span}}\, i'[0, u']$. Q. E. D.

ADDED IN PROOF. We have learned that Corollary 3 above is known (for separable Banach lattices) to H. P. Lotz and to H. Rosenthal.

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