BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

THE ROOTS OF A SIMPLE LIE ALGEBRA ARE LINEAR

BY ROBERT LEE WILSON[1]

Communicated by Barbara L. Osofsky, February 25, 1976

In this note we announce a result on the structure of a Cartan subalgebra of a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 7$. Consequences of this result include the linearity of the roots of a finite-dimensional simple Lie algebra over F and classification of the finite-dimensional simple Lie algebras over F whose roots generate a cyclic group.

1. **Structure of $\bar{H}$.** Let F be an algebraically closed field of characteristic $p > 7$. Let L be a finite-dimensional simple Lie algebra over F and let H be a Cartan subalgebra of L. Identify L with the subalgebra ad L of the restricted Lie algebra Der L. Let $\bar{H}$ denote the restricted subalgebra of Der L generated by H. Since $\bar{H}$ is restricted, each element of $\bar{H}$ has a Jordan-Chevalley decomposition into its semisimple and nilpotent parts ([3, §1] or [4, Chapter V.7]).

Now $\bar{H}$ is a nilpotent restricted Lie algebra and hence contains a unique maximal torus T. Also $\bar{H}$ contains a nil ideal I, called the nil radical, which contains every nil ideal. (An ideal is nil if every element in it is nilpotent.)

Our main result is

THEOREM 1. $\bar{H} = T + I$.

Since every root in the Cartan decomposition for L extends to a function on $\bar{H}$ which is linear on $T + I$ we have

COROLLARY. *Every root in the Cartan decomposition for L (with respect to H) is a linear function on H.*

The first result of this type is due to Schue [3], who proved $\bar{H} = T + I$ under the hypotheses that dim $T = 1$ and that every proper subalgebra of L is solvable.

To prove Theorem 1 (in [6]) we assume $\bar{H} \neq T + I$. Then (following Schue) we find an element $b \in \bar{H}$ such that $b^p \in I$, $[b, \bar{H}] \subseteq T + I$, and $[b, \bar{H}] \not\subseteq I$. We let $L = H + \Sigma_{\gamma \in \Gamma} L_\gamma$ be the Cartan decomposition of L with respect to H and set

$$S = \{(\gamma, \delta) \in \Gamma \times \Gamma \mid \gamma([b[L_\delta, L_{-\delta}]]) \neq (0)\}.$$

AMS (MOS) subject classifications (1970). Primary 17B20.

[1]This research was supported in part by National Science Foundation Grant Number MPS72-04547A03.

Using results of Schue we then show that either

(1) there exists $\alpha \in \Gamma$ with $(\alpha, \alpha) \in S$,

or

(2) for all $\gamma \in \Gamma$, $(\gamma, \gamma) \notin S$, but there exist $\alpha, \beta \in \Gamma$ with $(\alpha, \beta) \in S$ and $(\beta, \alpha) \in S$.

The major part of the proof is then devoted to showing that either (1) or (2) leads to a contradiction.

2. **Classification results.** Using Theorem 1 we prove (in [7])

THEOREM 2. *Let L be a finite-dimensional simple Lie algebra over F with Cartan decomposition $L = H + \Sigma_{\gamma\in\Gamma} L_\gamma$. If Γ generates a cyclic subgroup of H^* then L is isomorphic to* sl(2) *or to one of the simple Lie algebras of generalized Cartan type* $W(1\colon \mathbf{n})$ or $H(2\colon \mathbf{n}\colon \Phi)^{(\infty)}$.

(For the definition of the Lie algebras of generalized Cartan type see [5].)

The proof depends on showing the existence of a subalgebra $K \subseteq L + \bar{H}$ such that $K \supseteq \bar{H}$ and $[L, I] \subseteq K$. It is then possible to construct a filtration $L = L_{-1} \supseteq L_0 \supseteq L_1 \supseteq \cdots$ with L_0/L_1 either one-dimensional or isomorphic to sl(2). The theorem then follows from results of [5] characterizing the simple Lie algebras of generalized Cartan type.

The hypothesis of Theorem 2 that Γ generates a cyclic subgroup of H^* is easily seen to be equivalent to the requirement that dim $T = 1$. Thus Theorem 2 is similar to results of Kaplansky [2] and Block [1] on classification of simple Lie algebras with one-dimensional Cartan subalgebras.

REFERENCES

1. R. Block, *On Lie algebras of rank one*, Trans. Amer. Math. Soc. 112 (1964), 19–31 MR 28 #4013.

2. I. Kaplansky, *Lie algebras of characteristic p*, Trans. Amer. Math. Soc. 89 (1958), 149–183. MR 20 #5799.

3. J. R. Schue, *Cartan decompositions for Lie algebras of prime characteristic*, J. Algebra 11 (1969), 25–52; Errata 13 (1969), 558. MR 38 #201.

4. G. B. Seligman, *Modular Lie algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40, Springer-Verlag, Berlin and New York 1967. MR 39 #6933.

5. R. L. Wilson, *A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic*, J. Algebra (to appear).

6. ———, *Cartan subalgebras of simple Lie algebras*, Trans. Amer. Math. Soc. (submitted).

7. ———, *Simple Lie algebras of toral rank one*, Trans. Amer. Math. Soc. (submitted),

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903