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ON THE NUMBER OF SOLUTIONS TO PLATEAU'S PROBLEM

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Introduction. Since its formulation by Plateau in the 19th century, little (see [2], [4]) has been known about the number of simply connected minimal surfaces spanning a simple closed curve $\Gamma \subset R^3$. Existence was proved in the thirties by J. Douglas [1] and T. Radó [5]. In the paragraphs below we indicate how a new topological theory partially describes the way in which the number of minimal surfaces spanning a curve changes as the curve changes.

I. **Formulation of the problem.** Let $H^{r+2}(S^1, R^n)$ be the Sobolev Hilbert space of H^{r+2} maps the unit circle S^1 into R^n, with $r \geqslant 5$. Let $\mathcal{A} = \text{Emb}(S^1, R^3)$ be the open submanifold of $H^{r+2}(S^1, R^3)$ which consists of embeddings of S^1 into R^3. Let Γ be the image of such an embedding $\alpha \in \mathcal{A}$. Set η^α to be the component of $H^2(S^1, \Gamma)$ {the C^r Hilbert manifold of H^2 maps from S^1 to Γ} determined by the embedding α. Let $\mathcal{M}^\alpha$ be the open submanifold of η^α consisting of the diffeomorphisms. For every $u \in H^2(S^1, \Gamma) \subset H^2(S^1, R^3)$ we can extend $u = (u_1, \ldots, u_n)$ harmonically to the disc $\mathcal{D}$. Define the smooth energy functional $E_\alpha: \eta^\alpha \longrightarrow R$ by

$$E_\alpha(u) = \frac{1}{2}\sum_{i=1}^{3} \int_{\mathcal{D}} \left[\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2\right] dx\, dy.$$

Denote by $\overline{\mathcal{M}}^\alpha$ the closure of $\mathcal{M}^\alpha$ in η^α.

J. Douglas showed, in his pioneering work [1], that the critical points of E_α in $\overline{\mathcal{M}}^\alpha$ are simply connected minimal surfaces spanning Γ. We are interested in obtaining information on the number of critical points of E_α on $\overline{\mathcal{M}}^\alpha$.

II. **The theory.** Let M be a connected smooth Banach manifold and $K: T^2M \longrightarrow TM$ a connection map. In [6] the author defines a smooth vector field $X: M \longrightarrow TM$ to be Fredholm with respect to K if for each $p \in M$ the covariant derivative of X with respect to K, $\nabla X(p)$, which is a linear map of T_pM to itself, is linear Fredholm. By the *index of X* we mean the dim ker $\nabla X(p)$ − dim coker $\nabla X(p)$. A Fredholm vector field is Palais-Smale if $\nabla X(p)$ is of the form $I + C$, where C is a completely continuous linear map. Palais-Smale vector fields have index zero.

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Let X be a Palais-Smale vector field on M with finitely many isolated zeros in the interior of M. Then using the degree theory developed in [3] one can define the degree of X at a zero p, which we denote by $(\deg X)(p)$. The *Euler characteristic* $\chi(X)$ is defined to be

$$\chi(X) = \sum_{p \in \mathrm{zeros}(X)} (\deg X)(p).$$

If X has no zeros then $\chi(X) = 0$. By using elementary transversality techniques, the Euler characteristic can be defined for Palais-Smale vector fields with a compact set of zeros in the interior of M.

III. Applications.

THEOREM 1. *There exists a smooth connection K_α on the second tangent bundle $T^2\eta^\alpha$ and a smooth vector field X^α: $\eta^\alpha \longrightarrow T\eta^\alpha$ which is Palais-Smale with respect to the connection K_α and whose zeros are precisely all the critical points of E_α. Moreover, $X^\alpha(E_\alpha) = dE_\alpha(u)(X^\alpha(u)) \geqslant 0$.*

DEFINITION. Let $u \in \eta^\alpha$ be a minimal surface. A *branch point* $p \in \mathcal{D}$ of u is a point where the map u: $\mathcal{D} \longrightarrow \mathbf{R}^3$ fails to be an immersion. An embedding $\alpha \in \mathcal{A}$ is *fine* if all minimal surfaces spanning $\Gamma = \alpha(S^1)$ are free of branch points.

In [4] Radó showed that if α is not "too complicated" then α is fine. In particular, he showed that α is fine if there existed no point $q \in \mathbf{R}^3$ such that every hyperplane through q intersected Γ in at least four points.

THEOREM 2. *Let $\mathcal{F} \subset \mathcal{A}$ be the set of fine embeddings. Then $\mathcal{F}$ is open in $\mathcal{A}$, and hence open in $H^{r+2}(S^1, \mathbf{R}^3)$.*

CONJECTURE. $\mathcal{F}$ is dense in $\mathcal{A}$, or perhaps the open set of curves which admit no minimal surfaces with boundary branch points is dense in $\mathcal{A}$.

Let G be the three dimensional noncompact Lie group of bijective holomorphic maps of the disc onto itself. The functional E_α and the vector field X^α of Theorem 1 will be equivariant with respect to the action of G. Therefore, there is no hope that the critical points of E_α in $\overline{\mathcal{M}}^\alpha$ will be isolated since the orbit of any critical point will consist of critical points. Applying general transversality techniques we obtain

THEOREM 3. *For an open dense set of embeddings $\mathcal{V} \subset \mathcal{F}$ the zeros of X^α, $\alpha \in \mathcal{V}$, in $\mathcal{M}^\alpha$ are nondegenerate (and therefore isolated) three dimensional submanifolds of η^α. Moreover, for such $\alpha \in \mathcal{V}$ there are only finitely many such critical submanifolds.*

In general minimal surfaces on the same orbit are identified. Doing this we find

THEOREM 4. *If $\alpha \in \mathcal{V}$ and $\gamma \in \mathcal{F}$, $\gamma = \alpha + \rho$, is sufficiently close to α, then the minimal surfaces spanning γ are smooth functions of the parameter ρ.*

COROLLARY. *If $\alpha \in V$ and $\gamma \in F$ is sufficiently close to α, then the geometric number of minimal surfaces spanning γ is equal to the number spanning α.*

COROLLARY. *Any curve sufficiently close to a plane curve has a unique minimal surface spanning it.*

Let γ belong to F. We can define the Euler-characteristic of the corresponding vector field X^γ, and we take this to be the definition of the *algebraic number* of minimal surfaces spanning the image $\gamma(S^1)$.

Applying an Euler-Hopf theorem for Palais-Smale vector fields we get

THEOREM 5. *Let γ_0 and γ_1 be fine embeddings. Suppose further that γ_0 is isotopic to γ_1 through a family γ_t, $0 \leqslant t \leqslant 1$, of fine embeddings. Then the algebraic number of minimal surfaces spanning γ_0 is equal to the number spanning γ_1.*

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