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ADJOINT SEMIGROUP THEORY FOR A VOLTERRA INTEGRODIFFERENTIAL SYSTEM

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I. **Introduction.** In this note we announce some recent results concerning the semigroup theory for a class of linear Volterra integrodifferential systems. The system under consideration has previously been studied by Barbu and Grossman [2] and Miller [6], via semigroup methods. Although semigroup theory is employed in both of the above mentioned articles, it is important to note that the semigroup constructed in [6] differs greatly from the semigroup constructed in [2]. In particular, Miller is able to obtain certain stability results that do not hold for the semigroup constructed in [2]. However, we show that by an appropriate choice of the state space, Miller's semigroup may be considered as the "adjoint" semigroup (in the sense of Hille and Phillips [5]) to the semigroup constructed by Barbu and Grossman. We shall state the results without proofs. Proofs of the theorems will appear elsewhere (see [3]).

II. **Preliminaries.** If $x\colon (-\infty, 0] \longrightarrow C^n$ is given, then for $t \geqslant 0$, we define $x_t\colon (-\infty, 0] \longrightarrow C^n$ by $x_t(s) = x(t + s)$. For $1 \leqslant p \leqslant +\infty$, the usual Lebesgue space of C^n-valued functions on an interval with endpoints $-\infty \leqslant a < b \leqslant +\infty$ will be denoted by $L_p(a, b)$. Throughout this paper, M shall denote an $n \times n$ constant matrix and $K(\cdot)$ shall denote an $n \times n$ matrix function satisfying $\int_0^{+\infty} \|K(s)\|\, ds < +\infty$. Consider the linear Volterra integrodifferential equation,

$$x'(t) = Mx(t) + \int_{-\infty}^{t} K(t - s)x(s)\, ds, \tag{2.1}$$

with the initial data

$$x(0) = \eta, \qquad x_0(s) = \varphi(s) \quad \text{a.e. on } (-\infty, 0], \tag{2.2}$$

where $\eta \in C^n$ and $\varphi \in L_1(-\infty, 0)$.

A solution to system (2.1)–(2.2) is a function $x\colon (-\infty, +\infty) \longrightarrow C^n$ such that x is absolutely continuous (A.C.) on $[0, +\infty)$ and satisfies (2.1) a.e. on $[0, +\infty)$, $x(0) = \eta$, and $x_0(s) = \varphi(s)$ a.e. on $(-\infty, 0]$. We shall let $Z_1 = C^n \times L_1(-\infty, 0)$ denote the product space with the product norm. It can be shown

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that for each pair $(\eta, \varphi) \in Z_1$, there is a unique solution to (2.1) with initial data (2.2). Moreover, this solution depends continuously on the initial data (η, φ). For $t \geq 0$ define $S(t)\colon Z_1 \longrightarrow Z_1$ by $S(t)(\eta, \varphi) = (x(t), x_t(\cdot))$, where x is the unique solution of (2.1)–(2.2). It follows that $S(t)$ is a C_0 semigroup on Z_1. (Compare this semigroup with the one constructed in [2].)

THEOREM 1. *If* $\mathcal{A}$ *denotes the infinitesimal generator of* $S(t)$, *then*:
(i) $\mathcal{A}$ *is closed and densely defined*,
(ii) $\mathcal{D}(\mathcal{A}) = \{(\eta, \varphi) \in Z_1\colon \varphi$ *is A. C. on compact subsets of* $(-\infty, 0]$, $\varphi' \in L_1(-\infty, 0)$, $\varphi(0) = \eta\}$,
(iii) *if* $(\eta, \varphi) \in \mathcal{D}(\mathcal{A})$, *then* $\mathcal{A}(\eta, \varphi) = (M\eta + \int_{-\infty}^{0} K(-s)\varphi(s)\,ds, \varphi')$.

The adjoint space Z_1^* is seen to be $Z_\infty = C^n \times L_\infty(-\infty, 0)$. Moreover, by easy calculations we have the following result.

THEOREM 2. *If* $\mathcal{A}^*$ *denotes the adjoint of* $\mathcal{A}$, *then*:
(i) $\mathcal{D}(\mathcal{A}^*) = \{(\xi, \Psi) \in Z_\infty\colon \Psi$ *is A. C. on compact subsets of* $(-\infty, 0]$, *and* $\hat{\Psi} \in L_\infty(-\infty, 0)\}$, *where* $\hat{\Psi}(s) = K^*(-s)\xi - \Psi'(s)$,
(ii) *if* $(\xi, \Psi) \in \mathcal{D}(\mathcal{A}^*)$, *then* $\mathcal{A}^*(\xi, \Psi) = (M^*\xi + \Psi(0), \hat{\Psi})$,
(iii) $\mathcal{D}(\mathcal{A}^*)$ *is not dense in* Z_∞ *and* $\overline{\mathcal{D}(\mathcal{A}^*)} = Z_1^+ = \{(\xi, \Psi)\colon \Psi \in \mathrm{BUC}(-\infty, 0]\}$ *where* $\mathrm{BUC}(-\infty, 0]$ *is the set of all bounded, uniformly continuous functions defined on* $(-\infty, 0]$.

3. **The adjoint space.** The semigroup $S(t)$ generates a corresponding semigroup $S^*(t)$ on Z_1^*. However it is well known that $S^*(t)$ is not a C_0 semigroup, and in order to obtain the desired continuity properties for $S^*(t)$ it is necessary to restrict $S^*(t)$ to a proper subspace of Z_1^* (see [5]). Let $S^+(t)$ be the restriction of $S^*(t)$ to $Z_1^+ = \overline{\mathcal{D}(\mathcal{A}^*)}$. It is well known that $S^+(t)$ is a C_0 semigroup on Z_1^+ and the infinitesimal generator of $S^+(t)$ is the operator $\mathcal{A}^+$ defined to be $\mathcal{A}^*$ restricted to $\mathcal{D}(\mathcal{A}^+) = \{(\xi, \Psi) \in Z_1^+\colon \hat{\Psi}$ is in $\mathrm{BUC}(-\infty, 0]\}$. Consequently, if $(\xi, \varphi) \in \mathcal{D}(\mathcal{A}^+)$ then $\mathcal{A}^+(\xi, \varphi) = (M^*\xi + \Psi(0), \hat{\Psi})$, where $\hat{\Psi}(s) = K^*(-s)\xi - \Psi'(s)$.

This explicit representation of $\mathcal{A}^+$ provides the corresponding representation of $S^+(t)$.

THEOREM 3. *If* $(\xi, \Psi) \in Z_1^+$, *then* $S^+(t)(\xi, \psi) = (y(t), y^t(\cdot))$, *where* y *is the solution to the system*

$$y'(t) = M^*y(t) + \int_0^t K^*(t-s)y(s)\,ds + \Psi(-t) \tag{3.1}$$

with initial value

$$y(0) = \xi, \tag{3.2}$$

and $y^t(\cdot)\colon (-\infty, 0] \longrightarrow C^n$ *is defined by* $y^t(s) = \Psi(s-t) + \int_0^t K^*(t-s-v)y(v)\,dv$.

REMARK 1. We identify the space Z_1^+ with $W = \{(\xi, \Psi): \Psi \in \mathrm{BUC}[0, +\infty)\}$ by the following identification: for each function f: $(-\infty, 0] \longrightarrow C^n$ define F: $[0, +\infty) \longrightarrow C^n$ by $F(t) = f(-t)$, for $t \geqslant 0$. Consequently, we can identify $S^+(t)$ with a semigroup $U(t)$: $W \longrightarrow W$ and A^+ with the generator C of $U(t)$, in a natural way. In particular, if $(\xi, \Psi) \in \mathcal{D}(C)$, then $C(\xi, \Psi) = (M^*\xi + \Psi(0), \hat{\Psi})$, where $\hat{\Psi}$: $[0, +\infty) \longrightarrow C^n$ is now defined by $\hat{\Psi}(t) = K^*(t)\xi + \Psi'(t)$.

REMARK 2. If F: $(-\infty, 0] \longrightarrow C^n$ is a bounded continuous function, then $L(F) = f$ is the bounded uniformly continuous function defined on $[0, +\infty)$ by

$$[L(F)](t) = f(t) = \int_{-\infty}^{0} K^*(t-s)F(s)\, ds.$$

Let $Y(K^*)$ be the subspace of W given by $Y(K^*) = \{(\xi, f): f = L(F), \xi = F(0), F \text{ is bounded and continuous on } (-\infty, 0]\}$, and the closure of $Y(K^*)$ in W will be denoted by Y.

It is now clear that Miller's semigroup (defined on Y) is the semigroup $U(t)$ restricted to Y. That is, Miller's semigroup can be considered as the restriction of the "adjoint" semigroup (under the identification of W with Z_1^+) of $S(t)$. This is a useful interpretation of $U(t)$ since we may now make certain inferences concerning the relationship between the stability of $U(t)$ and the stability of solutions of (2.1)–(2.2).

As a simple example consider the following.

COROLLARY 1. *Let* $(\eta, \varphi) \in Z_1$ *and* x *be the solution of* (2.1)–(2.2). *If* $\{\lambda: \operatorname{Re} \lambda \geqslant 0, \det[M^* - \int_{-\infty}^{0} K^*(-s)e^{\lambda s}\, ds - \lambda I] = 0\}$ *is empty, then* $\|x(t)\| \longrightarrow 0$ *as* $t \longrightarrow +\infty$.

REMARK 3. In [7], Miller considers the system (2.1)–(2.2) in a Banach space X, and under certain assumptions points out the direct embedding of his semigroup in $U(t)$. Again it is easy to check that this can be considered as the adjoint semigroup of $S(t)$ where $S(t)$ is defined on $X \times L_1((-\infty, 0], X)$.

All of the above results are special cases of general theorems for a class of linear functional differential equations with infinite delays (see [3]). For a treatment of $S(t)$ in the case of functional differential equations with finite delays, see Borisovič and Turbabin [4], Banks and Burns [1].

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