BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

INJECTIVE STABILITY FOR K_2 OF LOCAL RINGS

BY R. KEITH DENNIS[1] AND MICHAEL R. STEIN[2]

Communicated by Hyman Bass, April 3, 1974

It has been conjectured for some time **[St2]**, **[D]**, **[S-D]**, **[D-S2]** that for a semilocal ring R, the homomorphisms $\theta_n: K_2(n, R) \to K_2(n+1, R)$, known to be surjective for all $n \geqq 2$ **[St2]**, **[S-D]**, are in fact isomorphisms for $n \geqq 3$. Various special cases have been proved, most notably the difficult theorem of Matsumoto for fields [**Ma**, Corollaire 5.11] and the case of discrete valuation rings **[D-S1]**. Matsumoto also shows that θ_2 is not an isomorphism in general.

In this note we announce the proof of the following theorem, details of which will be published elsewhere. Unexplained notation and terminology is that of **[Mi]**.

THEOREM A. *Let R be a commutative local ring. The homomorphisms θ_n are isomorphisms, and consequently, $K_2(n, R) \approx K_2(R)$ for any $n \geqq 3$.*

In broad outline, the proof of Theorem A is similar to that of Matsumoto for fields [**Mi**, §12]. The maps θ_n are surjective as $K_2(n, R)$ is generated by the Steinberg symbols $\{u, v\}$, $u, v \in R^*$ [**St2**, Theorem 2.13]. To show that for $n \geqq 3$ the θ_n are injective, the symbol $\{\ ,\ \}$ with values in the group $A = K_2(3, R)$ is used to construct a central extension

$$(*) \qquad 1 \to A \to G_n \to E(n, R) \to 1.$$

Since $\mathrm{St}(n, R)$ is the universal central extension of $E(n, R)$ for $n \geqq 5$, $(*)$ implies the existence of a surjective homomorphism $\mathrm{St}(n, R) \to G_n$ which induces a surjection $K_2(n, R) \to A$ inverse to the surjection

$$K_2(3, R) \to K_2(n, R).$$

Thus Theorem A follows immediately from the construction of the central extensions $(*)$.

AMS (MOS) subject classifications (1970). Primary 18F25, 20G35; Secondary 20G10.
Key words and phrases. K_2, local rings, radical ideal, stability theorems for K_2, generators and relations for K_2, Bruhat decomposition.
[1] Partially supported by NSF GP 25600.
[2] Partially supported by NSF GP 28915.

As in [**Mi**, Lemmas 12.1–12.5], we begin by constructing a central extension

$$1 \to A \to W_n \to M_n \to 1$$

where M_n is the group of monomial matrices in $E(n, R)$. By [**Mi**, Lemma 12.6], for any field F there exists a well-defined retraction $\rho: E(n, F) \to M_n$. If R is a local ring and $E \in E(n, R)$, we denote by $\pi(E)$ the permutation associated to the monomial matrix $\rho(\bar{E})$ over the residue field of R. We replace the Bruhat form used by Matsumoto (it exists if and only if the ring in question is a skew-field) by the form given in the following

LEMMA. *Let R be a local ring. Then every matrix in $E(n, R)$ can be written uniquely as a product ULW, where U is upper triangular, L is lower triangular and $W \in M_n$ satisfies $\pi(ULW) = \pi(W)$.*

Thus we define a retraction $\rho: E(n, R) \to M_n$; we then let

$$X_n \subset W_n \times E(n, R)$$

denote the set of pairs having the same associated monomial matrix. The group G_n of (*) is constructed as a group of permutations acting on the left of X_n. It is automatic that G_n acts transitively; the most difficult part of the proof is to show that this action is simply transitive. In the case of a field, Matsumoto proves this by defining a second group of permutations acting transitively on the right of X_n; the major computation in his paper shows that the two group actions commute. Due to the lack of symmetry in our normal form, a proof of this type presents complexities increasing in difficulty with n.

To avoid this problem, we use the following generalization of a theorem of Curtis and Tits [**C**, Theorem (1.4)], [**T**], which is of interest in its own right. The notation is that of [**St1**, §3].

THEOREM B. *Let Φ be a reduced irreducible root system of rank $n \geqq 2$ with $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a (fixed) system of simple roots in Φ. For each pair of integers k, l, $1 \leqq k, l \leqq n$, denote by Φ_{kl} the subset of Φ spanned by α_k and α_l.*

Let R be a commutative ring with 1, and define $\Sigma(\Phi, R)$ to be the group with generators $\xi_\alpha(t)$, $\alpha \in \bigcup \Phi_{kl}$, $t \in R$, subject to the relations

$$\text{(P1)} \qquad \xi_\alpha(s)\xi_\alpha(t) = \xi_\alpha(s+t), \qquad \alpha \in \bigcup \Phi_{kl}, \quad s, t \in R,$$

$$\text{(P2)} \qquad [\xi_\alpha(s), \xi_\beta(t)] = \prod \xi_{i\alpha+j\beta}(N_{\alpha\beta ij} s^i t^j)$$

for all α, $\beta \in \Phi_{kl}$ for some $1 \leqq k, l \leqq n$, such that $\alpha + \beta \neq 0$, where the product and the integers $N_{\alpha\beta ij}$ are as in [**St1**, 3.7, (**R2**)]. (*Note that we assume* (P2) *only when α and β lie in the same Φ_{kl}.*)

Then the natural surjection $p:\Sigma(\Phi, R)\to \mathrm{St}(\Phi, R)$ *defined by*

$$p(\xi_\alpha(t)) = x_\alpha(t)$$

is an isomorphism.

For $\Phi=A_n$, the Steinberg group $\mathrm{St}(\Phi, R)=\mathrm{St}(n+1, R)$ is also defined for noncommutative rings. In this case we obtain

THEOREM B′. *Let R be any ring. For* $n\geqq 3$, $\mathrm{St}(n, R)$ *can be presented by generators* $x_{ij}(r)$, $r\in R$, $i\neq j$, $|i-j|\leqq 2$, *subject to those Steinberg relations involving only three consecutive indices together with the relations* $[x_\alpha(r), x_\beta(s)]=1$, *where* $\pm\alpha=(i, i+1)$, $\pm\beta=(j, j+1)$, $i+1<j$.

Roughly speaking, this theorem allows one to define a homomorphism $\mathrm{St}(n, R)\to G_n$ once the action of G_n on X_n has been determined in each "3×3 block".

A presentation for $K_2(3, R)$ (and hence for $K_2(n, R)$ and $K_2(R)$) can be obtained by explicitly carrying out the computations to determine what relations are forced on $\{\ ,\ \}$ in order for the action of G_3 on X_3 to be simply transitive. Due to the complexity of these relations (there are eleven distinct families), we do not list them in this note; they will appear in the detailed account of the proof.

By a similar technique, using the LHU form of [**St2**, Theorem 2.4], we prove

THEOREM C. *Let J be any ideal contained in the Jacobson radical of the commutative ring R. Then the homomorphisms* $K_2(n, R, J)\to K_2(n+1, R, J)$ *are isomorphisms for all* $n\geqq 3$.

Here the relative group $K_2(n, R, J)$ is that defined in [**Mi**, p. 54] or [**St3**, Definition 1.3]; it is *not* the kernel of the map $K_2(n, R)\to K_2(n, R/J)$. Hence Theorem C is not known to imply Theorem A for semilocal rings. In case the homomorphism $R\to R/J$ splits, Theorem A for semilocal rings can be deduced from Theorem C using a result of Swan [**Sw**, Corollary 7.3]. As in the case of Theorem A, defining relations for these groups are also obtained.

REFERENCES

[C] C. W. Curtis, *Central extensions of groups of Lie type*, J. Reine Angew. Math. **220** (1965), 174–185. MR **32** #5738.

[D] R. K. Dennis, *Stability for* K_2, Proc. Conference on Orders, Group Rings and Related Topics, Lecture Notes in Math., vol. 353, Springer-Verlag, Berlin and New York, 1973, pp. 85–94.

[D-S1] R. K. Dennis and M. R. Stein, K_2 *of discrete valuation rings*, Advances in Math. (to appear).

[D-S2] ———, *The functor* K_2: *a survey of computations and problems*, Algebraic K-Theory II, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973, pp. 243–280.

[Ma] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62. MR **39** #1566.

[Mi] J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Studies, no. 72, Princeton Univ. Press, Princeton, N.J., 1971.

[S-D] M. R. Stein and R. K. Dennis, *K_2 of radical ideals and semi-local rings revisited*, Algebraic *K*-theory II, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973, pp. 281–303.

[St1] M. R. Stein, *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math. **93** (1971), 965–1004.

[St2] ———, *Surjective stability in dimension* 0 *for K_2 and related functors*, Trans. Amer. Math. Soc. **178** (1973), 165–191.

[St3] ———, *Relativizing functors on rings and algebraic K-theory*, J. Algebra **19** (1971), 140–152. MR **44** #288.

[Sw] R. G. Swan, *Excision in algebraic K-theory*, J. Pure Appl. Algebra **1** (1971), no. 3, 221–252. MR **46** #7354.

[T] J. Tits, unpublished; see [C, footnote 3].

Department of Mathematics, Cornell University, Ithaca, New York 14850

Department of Mathematics, Northwestern University, Evanston, Illinois 60201