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BEST APPROXIMATION IN $L^1(T)$

BY J.-P. KAHANE[1]

0. The address I gave in Minneapolis was entitled *Hereditary properties of metric projections*. The general topic is explained in §§1 and 2, together with some recent examples. The subsequent sections develop the last part of the talk concerning $L^1(\boldsymbol{T})$.

§1 consists of general results on metric projection from a Banach space X to a closed subspace Y; that is, best approximation of each $x \in X$ by a unique $y \in Y$. We are interested in the case where such an operator exists (in general, it is not linear).

§2 deals with hereditary properties (in other words, preservation of classes of functions) when $X=L^p(\boldsymbol{T})$ and Y is a closed subspace invariant under translation. Several classes of functions are considered. The question originated with H. S. Shapiro (1952); recent contributions are due to Adamyan, Arov and Krein (1969), Carleson and Jacobs (1972), H. S. Shapiro (1973) and myself (1973).

§3 considers the case $X=L^1(\boldsymbol{T})$; it is taken from **[12]**.

§§4, 5, and 6 develop **[13]**. §4 deals with the metric projection from L^1 to H^1 (already defined by Doob in 1941). The oldest result—and maybe the best—on hereditary properties in this case goes back to F. Riesz (1920): trigonometric polynomials are mapped into trigonometric polynomials. Preservation of properties by Toeplitz operators is considered, and some precise results are obtained on the classes defined in §2. In relation with division of analytic functions, this topic has been considered by B. I. Korenbljum and V. S. Korolevič (1970), B. I. Korenbljum (1971), F. A. Šamoyan (1971), V. P. Havin (1971), M. Rabindranathan (1972), B. I. Korenbljum and V. M. Faivyševskiĭ (1972), V. E. Katznelson (1972), N. A. Širokov (1972), and E. M. Dyn'kin (1973).

§5 deals with the metric projection from L^1 to the subspace of $1/2\nu+1$-periodic functions in L^1, a topic already considered by Steiner (1837); the subject belongs to elementary geometry and most questions are open when $\nu \geqq 2$.

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§6 deals with the metric projection from L^1 to the subspace of $1/2\nu+1$-periodic functions in H^1 and contains only very partial results.

1. **Metric projections.** Here is the general setting. X is a metric space, Y a subset of X. For each $x \in X$ we consider the set

$$\mathscr{P}(x) = \{y \in Y \mid d(x, y) = d(x, Y)\},$$

that is, the subset of Y (maybe empty) where the function $d(x, \cdot)$ attains its infimum. Some authors call $\mathscr{P}(x)$ the metric projection of x on Y. The elements of $\mathscr{P}(x)$ are the elements of best approximation to x in Y. In this paper we are interested in cases where $\mathscr{P}(x)$ consists of exactly one point for each $x \in X$. This point (the unique element of best approximation to x in Y) will be denoted by Px. We consider the mapping P from X to Y, and that is what we now call metric projection.

We restrict outselves to X a Banach space and Y a closed subspace. Here are the most classical examples.

(a) $X = C_r(I)$, the space of real-valued continuous functions on a compact interval I, and Y consists of all polynomials of degree $\leqq n$. This is the case considered and beautifully worked out by Čebyšev.[2]

(b) X is a Hilbert space. Then P is the ordinary (linear) projection.

(c) X is uniformly convex (=uniformly rotund). Then P exists and is continuous. This holds for $X = L^p(\mu)$ (μ is a measure, and $1 < p < \infty$ *strict* inequalities).

(d) When X is not uniformly convex, it may happen that $\mathscr{P}(x)$ is empty or contains more than one point for some x. It may also happen that P exists and is not continuous (Lindenstrauss, see **[27]**).

Duality methods in best approximation were introduced about 25 years ago and proved very useful (see **[2]** for a history of the topic, **[9]** and **[27]** for applications). The main tools are the following formulae, simple consequences of the Hahn-Banach theorem.

Let X^* denote the dual space of X; we have

$$d(x, Y) = \inf_y \|x - y\| = \max_u |(u, x)|, \tag{1.1}$$

where $y \in Y$, u belongs to the unit sphere of X^* and $u \perp Y$, and max means that the supremum is reached.

If $X = \Xi^*$, the dual of some Banach space Ξ,

$$d(x, Y) = \min_y \|x - y\| = \sup_u |(u, x)|, \tag{1.2}$$

[2] Let us remark that the work of Čebyšev begins in 1853, while the Weierstrass theorem on polynomial approximation is dated 1885. The theory of best approximation happens to be older than the usual approximation theory. Let us remark also that Čebyšev deals with the *difficult* case (X not uniformly convex).

where $y \in Y$, u belongs to the unit sphere of Ξ and $u \perp Y$, and min means that the infimum is reached.

Formula (1) is used when $X=L^p(\mu)$ $(1 \leqq p < \infty)$; then $X^* = L^q(\mu)$ with $(1/p)+(1/q)=1$. Formula (2) is used when $X=L^\infty(\mu)$ with $\Xi = L^1(\mu)$. The *dual problem* is to study the set of u where the supremum is attained.

From now on X will consist of (classes of) functions defined on $\boldsymbol{T}$, the circle equipped with the Haar (=Lebesgue normalized) measure. For each $x \in X$, we write the Fourier series of x

$$x \sim \sum_{-\infty}^{\infty} \hat{x}(n) e_n \qquad (e_n(t) = \exp 2\pi int).$$

Let us describe the situation for $X=L^p(\boldsymbol{T})$ and $Y=$the Hardy class H^p, i.e. the subspace of $L^p(\boldsymbol{T})$ which consists of all functions y with $\hat{y}(n)=0$ for $n<0$. If $p=2$, P is the mapping $x \to y$ defined by

$$y \sim \sum_{0}^{\infty} \hat{x}(n) e_n.$$

If $p=1$, P exists and is continuous; the existence had been proved by Doob already in 1941 **[8]**, and the continuity derives from the "pseudo uniform convexity" of H^1, proved by D. J. Newman in 1963 **[19]**. If $p=\infty$, $\mathscr{P}x$ is never empty but may contain several points; $\mathscr{P}x$ consists of only one point if x is a continuous function ($x \in C$) and therefore also if $x \in C+H^\infty$; moreover the dual problem has solutions (that is, there exist some u where the supremum in (2) is attained) **[22]**, **[5]**. Since $C+H^\infty$ is a Banach space,[3] the metric projection from $X=C+H^\infty$ to H^∞ exists. If $p \neq 1, 2, \infty$, P exists and is continuous.

2. **Hereditary properties.** Suppose P exists. We can look for properties which are preserved by P; that is, for subsets A of X such that $PA \subset A$. We call such a property hereditary for P. Apparently the question was raised by H. S. Shapiro in his thesis (1952); he proved that analyticity is hereditary when $X=C+H^\infty$, $Y=H^\infty$ and when $X=L^1(\boldsymbol{T})$, $Y=H^1$.

Working with $X=L^p(\boldsymbol{T})$ $(1 \leqq p < \infty)$, we shall consider only closed subspaces invariant under translation. Therefore $Y=L^p_\Lambda(\boldsymbol{T})$, the closed subspace of $L^p(\boldsymbol{T})$ which consists of all y of the form $y \sim \sum_{\lambda \in \Lambda} \hat{y}(\lambda) e_\lambda$ where Λ is some subset of $\boldsymbol{Z}$ (set of all integers). In the case $X=C+H^\infty$, we shall be satisfied with $Y=H^\infty$.

Already the case $p=2$ has some interest. Here the projection $P: x \to y$ is defined by

$$y \sim \sum_{\lambda \in \Lambda} \hat{x}(\lambda) e_\lambda,$$

[3] See **[24]**. Actually $C+H^\infty$ is a Banach algebra.

that is, $\hat{y}=\hat{x}1_\Lambda$ (1_Λ is the indicator function of Λ defined on $\boldsymbol{Z}$). A subclass of $L^2(\boldsymbol{T})$ defines an hereditary property if and only if 1_Λ is a "multiplier" of this class.

Here are the classes we shall consider in connection with the metric projection from L^p to L^p_Λ (from now on we drop $\boldsymbol{T}$).

—L^r (with $r>p$);

—C (continuous functions);

—Λ_ω, that is the class of all continuous functions f such that

$$\sup_t |f(t+h)-f(t)| = O(\omega(h)) \qquad (h\to 0), \tag{2.1}$$

where $\omega(h)$ is a positive concave function of $h>0$, with $\lim_{h\to 0}\omega(h)=0$;

—Λ_α (Lipschitz class of order α), that is Λ_ω when $\omega(h)=h^\alpha$ $(0<\alpha<1)$;

—Λ_* (Zygmund class) consisting of all continuous f such that

$$\sup_t |f(t+h)+f(t-h)-2f(t)| = O(h) \qquad (h\to 0); \tag{2.2}$$

—Λ^∞_α (the notation will be clear in a moment) is Λ_α when $0<\alpha<1$, Λ_* when $\alpha=1$, and consists of all functions f having continuous derivatives $f',\cdots,f^{(\nu)}$ with $f^{(\nu)}\in\Lambda^\infty_{\alpha'}$ when $\alpha=\nu+\alpha'$, ν an integer and $0<\alpha'\leqq 1$;

—Λ^r_ω, Λ^r_α obtained by considering functions in L^r instead of continuous functions, and norms in L^r instead of sup-norms in (2.1) and (2.2):

$f\in\Lambda^r_\omega$ means $\|f(\cdot+h)-f(\cdot)\|_r=O(\omega(h))$;

$f\in\Lambda^r_\alpha$ $(0<\alpha<1)$ means $\|f(\cdot+h)-f(\cdot)\|_r=O(h^\alpha)$;

$f\in\Lambda^r_1$ means $\|f(\cdot+h)+f(\cdot-h)-2f(\cdot)\|_r=O(h)$;

$f\in\Lambda^r_{\nu+\alpha'}$ means $f^{(\nu)}\in\Lambda^r_{\alpha'}$, $f^{(\nu)}$ defined as a distribution;

—C^∞ (infinitely differentiable functions);

—$C(\{M_n\})$, class of all $f\in C^\infty$ such that $\log\|f^{(n)}\|_\infty\leqq\log M_n+O(n)$, where $\{M_n\}$ is a given positive sequence. For example, $C(\{n!\})$ is the analytic class, and $C(\{(n!)^\gamma\})$ $(\gamma>1)$ is called the Gevrey class of index γ. We shall restrict ourselves to the case

$$\log M_{n+1} = \log M_n + O(n); \tag{2.3}$$

then $C(\{M_n\})$ is closed under differentiation, and

$$f\in C(\{M_n\}) \Leftrightarrow \log\|f^{(n)}\|_2 \leqq \log M_n + O(n) \tag{2.4}$$

(because $\|f^{(n)}\|_2\leqq\|f^{(n)}\|_\infty\leqq\|f^{(n)}\|_2+c\|f^{(n+1)}\|_2$).

—$\mathscr{F}l^2(\rho)$, where ρ stands for a positive increasing sequence $\{\rho_n\}$ $(n\geqq 0)$ extended by $\rho_{-n}=\rho_n$ for $n<0$; by definition

$$f\in\mathscr{F}l^2(\rho) \quad \text{means} \quad \sum|\hat{f}(n)|^2\rho_n<\infty.$$

Before going further, let us mention the relation between the classes Λ^r_ω, Λ^r_α and the best approximation by trigonometric polynomials in the L^r-norm ($1\leqq r\leqq\infty$). We write $\mathscr{T}_n$ for the class of all trigonometric polynomials of order $\leqq 2^n$, and V_n for the de la Vallée Poussin kernel defined by $\hat{V}_n=1$ on $[-2^n, 2^n]$, $\hat{V}_n=0$ outside $[-2^{n+1}, 2^{n+1}]$, and $\hat{V}_n$ linear on each of the intervals $[-2^{n+1}, -2^n]$, $[2^n, 2^{n+1}]$. Finally we write $\omega_n=\omega(2^{-n})$ and, if the condition

$$\sum_1^\infty \omega_n < \infty \tag{2.5}$$

is fulfilled, ω^* will denote any concave function such that

$$\omega_n^* \geqq \sum_{j=0}^n 2^{-j}\omega_{n-j} + \sum_{j=0}^\infty \omega_{n+j}. \tag{2.6}$$

The following implications hold:

$$\text{(a)} \Rightarrow \text{(b)} \Leftrightarrow \text{(c)} \Rightarrow \text{(d)} \Leftrightarrow \text{(e)} \Rightarrow \text{(f)}, \tag{2.7}$$

where (a), (b), (c), (d), (e), and (f) are defined by:

(a) $f\in\Lambda^r_\omega$,
(b) $\inf_{p\in\mathscr{T}_n} \|f-p\|_r=O(\omega_n)$,
(c) $\|f-V_n*f\|_r=O(\omega_n)$,
(d) $\|(V_{n+1}-V_n)*f\|_r=O(\omega_n)$,
(e) $f=\sum_{n=1}^\infty f_n$, $\|f_n\|_r=O(\omega_n)$, $f_n\in\mathscr{T}_{n+1}$ and $\hat{f}_n=0$ on $[-2^{n-1}, 2^{n-1}]$,
(f) (if (2.5) holds $f\in\Lambda^r_{\omega^*}$.

Though this is well known, let us sketch the proof: (a)$\Rightarrow$(b) by taking Jackson sums (the Jackson kernel is the square of the Fejer kernel up to a multiplicative constant). (b)$\Rightarrow$(c) because $V_n*p=p$ and $\|V_n*(f-p)\|_r\leqq \|V_n\|_1\|f-p\|_r\leqq 3\|f-p\|_r$. (c)$\Rightarrow$(b) because $V_n*f\in\mathscr{T}_{n+1}$ and $\omega_n\leqq 2\omega_{n+1}$. (c)$\Rightarrow$(d) is obvious. (d)$\Rightarrow$(e) by taking $f_n=(V_n-V_{n-1})*f$. (e)$\Rightarrow$(d) because $(V_{n+1}-V_n)*f=(V_{n+1}-V_n)*(f_n+f_{n+1}+f_{n+2})$. (e)$\Rightarrow$(f) by S. Bernstein's theorem on derivatives of trigonometric polynomials, using

$$\|f_n(\cdot+2^{-\nu})-f_n(\cdot)\|_r \leqq 2^{-\nu}2^{n+1}\|f_n\|_r \quad \text{for } n\leqq\nu,$$

$$\|f_n(\cdot+2^{-\nu})-f_n(\cdot)\|_r \leqq 2\|f_n\|_r \qquad \text{for } n\geqq\nu.$$

For Λ^r_α the situation is simpler, and we have the equivalence

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)}, \tag{2.8}$$

(a) $f\in\Lambda^r_\alpha$,
(b) $\inf_{p\in\mathscr{T}_n} \|f-p\|_r=O(2^{-n\alpha})$,
(c) $\|f-V_n*f\|_r=O(2^{-n\alpha})$,
(d) $\|(V_{n+1}-V_n)*f\|_r=O(2^{-n\alpha})$,
(e) $f=\sum_{n=1}^\infty f_n$, $\|f_n\|_r=O(2^{-n\alpha})$, $f_n\in\mathscr{T}_{n+1}$, $\hat{f}_n=0$ on $[-2^{n-1}, 2^{n-1}]$.

This holds for $1 \leqq r \leqq \infty$ and $0<\alpha<\infty$. The proof is still easier. Another equivalent definition is given in [26]. In the case where $r=\infty$ and $0<\alpha<1$, (a)$\Rightarrow$(b) is due to Jackson, (b)$\Rightarrow$(a) to S. Bernstein; for $\alpha=1$, (a)$\Leftrightarrow$(b) is due to Zygmund.

The problems of heredity of properties we consider are the following: (1) Given $\Lambda \subset \boldsymbol{Z}$, $1 \leqq p \leqq \infty$, such that the metric projection P of L^p on L^p_Λ exists, which are the above classes preserved by P? (2) In case a class is not preserved, into which larger class is it mapped?

In the next sections these problems will be handled in detail for $p=1$.

The case $p=2$. Let us look briefly at the situation for $p=2$; then P exists and is linear for all Λ.

(2.9) *All classes* Λ^2_α, $C(\{M_n\})$, $\mathscr{F}l^2(\rho)$ *are preserved by* P *whatever* Λ *may be* [26].

(2.10) C *is preserved by* P *if and only if* Λ *is a finite union of cosets of subgroups of* $\boldsymbol{Z}$ *up to a finite set.* This is a theorem of Helson [11].

(2.11) Λ^∞_α *is preserved by* P *if and only if*

$$\left\|\sum_{\lambda\in\Lambda}(\hat{V}_{n+1}(\lambda) - \hat{V}_n(\lambda))e_\lambda\right\|_1 = O(1) \qquad (n \to \infty) \qquad [29].$$

The same holds for Λ^1_α. The condition is satisfied if the intersection of Λ with each dyadic interval $[-2^{n+1}, -2^n]$ or $[2^n, 2^{n+1}]$ is the difference $A\backslash B$ of two sets A and B $(A \supset B)$, and both A and B are disjoint unions of at most K cosets of subgroups of $\boldsymbol{Z}$, where $K=K(\Lambda)$ does not depend on n. It is not clear whether this is necessary as well as sufficient.

(2.12) L^r $(2<r<\infty)$ *is preserved whenever* Λ *is a union of dyadic blocks of the form* $[-2^{n+1}, -2^n]$ *or* $[2^n, 2^{n+1}]$ (Littlewood-Paley, see [30]). This is far from necessary.

(2.13) *In case* $\Lambda=\boldsymbol{N}=\{0, 1, 2, \cdots\}$, *we have the projection of* L^2 *on* H^2, *and the situation is well known*: *all classes* L^r $(2<r<\infty)$, Δ^r_α $(1\leqq r\leqq\infty)$, $\alpha>0$, $C(\{M_n\})$, $\mathscr{F}l^2(\rho)$ *are preserved*; L^∞ *and* C *are not preserved*; Λ_ω *is mapped into* C *if* (2.5) *holds, and this is a best condition*; *moreover*, Λ_ω *is mapped into* Λ_{ω^*}.

The case $p=\infty$. Let us consider now $X=C+H^\infty$, $Y=H^\infty$. The following theorem is due to Carleson and Jacobs [3] (H. S. Shapiro [25] for $C(\{n!\})$; see also Adamyan, Arov and Krein [1]).

(2.14) *The classes* Λ^∞_α *are preserved* (*proved for* α *not integral*, *probably also for* α *integral*). *If*, *in addition to* (2.3), *the sequence* $(1/n)\log M_n - \log n$ *is increasing*, *the class* $C(\{M_n\})$ *is preserved. C is not preserved. If* (2.5) *holds*, Λ_ω *is mapped into* C, *and this is a best condition.*

The proof uses properties of analytic functions in a quite elaborate way.

The case $p\neq 1, 2, \infty$. The metric projection exists for each Λ. There are essentially three positive results on heredity.

(2.15) *Whatever* Λ, *the class* Λ_α^p *is mapped into* $\Lambda_{\alpha/p}^p$ *if* $0<\alpha\leqq 2$ *and* $1<p<2$, *and into* $\Lambda_{\alpha/p'}^p$ *if* $0<\alpha\leqq 2$ *and* $p>2$, $1/p+1/p'=1$.

(2.16) *If* $\Lambda=N$ (*metric projection of* L^p *on* H^p), *and* $r>p$, L^r *is preserved.*

(2.17) *If* $\Lambda=N$, *analytic functions are mapped onto piecewise analytic functions.* (2.15) is due to H. S. Shapiro **[26]**. (2.16) is due to L. Carleson [3]. (2.17) (for rational functions) was stated in **[18]** and proved in **[22]**.

The main negative result (personal communication of H. S. Shapiro) is

(2.18) *If* $\Lambda=N$, *there exists a trigonometric polynomial* $x=ae_{-2}+be_{-1}$ *such that* $y=x+e_{-2}(1+e_1)^{2/p}$.

Therefore, there is no hope of extending (2.13) or (2.14) if $2<p<\infty$ and generally when $2/p$ is not integral.

The case $p=1$. We shall see later that we have only to consider the cases $\Lambda=N$, $\Lambda=(2p+1)Z$ (p integer) and $\Lambda=(2p+1)N$. The answer to the question whether a given class is preserved under the metric projection from L^1 to L_Λ^1 is given in the following table:

	$\Lambda=N$	$\Lambda=3Z$	$\Lambda=5Z, 7Z, \cdots$	$\Lambda=3N, 5N, \cdots$
L^r $(1<r<\infty)$	yes	yes	yes	yes
C, L^∞	no	yes	yes	no
Λ_α $(0<\alpha<1)$	yes	yes	no	?
Λ_α $(\alpha\geqq 1)$	yes	no	no	?
$C(\{M_n\})$	yes	no	no	?

The following statements will be proved in the next sections.

(2.19) *In case* $\Lambda=N$, *all classes* L^r $(1<r<\infty)$, Λ_α^r $(1\leqq r\leqq\infty)$, $C(\{M_n\})$, $\mathscr{F}l^2(\rho)$ *are preserved*; *the class of all trigonometric polynomials of order*

$\leqq N$ (N given) is preserved; L^∞ and C are not preserved; Λ_ω is mapped into C if (2.5) holds, and this is a best condition; moreover, Λ_ω is mapped into $\Lambda_{\omega *}$; more generally, Λ^r_ω is mapped into $\Lambda^r_{\omega *}$ ($1 \leqq r \leqq \infty$).

(2.20) *In case $\Lambda = (2\nu+1)\boldsymbol{Z}$, L^r, C, L^∞ are preserved, Λ_α ($\alpha \geqq 1$) is not preserved, Λ_α ($0<\alpha<1$) is preserved when $\nu=1$, Λ_α ($0<\alpha<1$) is mapped into $\Lambda_{\alpha/2}$ when $\nu \geqq 2$.*

(2.21) *In case $\Lambda = (2\nu+1)\boldsymbol{N}$, L^r ($1<r<\infty$) is preserved, L^∞ and C are not preserved.*

3. **Metric projection from L^1 to L^1_Λ.** In order to avoid trivialities, we suppose $\Lambda \neq \varnothing$, $\Lambda \neq \boldsymbol{Z}$.

We have the following characterization.

(3.1) *The metric projection from L^1 to L^1_Λ exists if and only if Λ is an infinite arithmetical progression with an odd difference.*

In other words, Λ is obtained from one of the typical cases $\Lambda = \boldsymbol{N}$, $\Lambda = (2\nu+1)\boldsymbol{Z}$, $\Lambda = (2\nu+1)\boldsymbol{N}$ using translation or symmetry.

Proof of (3.1). We use (1.1) with $X = L^1$, $Y = L^1_\Lambda$, $X^* = L^\infty$. We write Γ for the set of $x \in X$ such that $\|x\| = d(x, Y)$:

$$x \in \Gamma \Leftrightarrow \exists u \in L^\infty, \qquad u \perp L^1_\Lambda, \qquad \|u\|_\infty = 1, \qquad (u, x) = \|x\|.$$

Let us write $\Lambda' = \boldsymbol{Z} \backslash \Lambda$ and $\tilde{\Lambda}' = -\Lambda'$; then $u \in L^\infty$, $u \perp L^1_\Lambda$ becomes $u \in L^\infty_{\tilde{\Lambda}'}$. On the other hand $(u, x) = \|x\|$ means $ux = |x|$ almost everywhere. Therefore

$$(3.2) \qquad x \in \Gamma \Leftrightarrow \exists u \in L^\infty_{\tilde{\Lambda}'}, \qquad \|u\|_\infty = 1, \qquad ux = |x| \quad \text{a.e.}$$

Uniqueness holds (that is, card $\mathscr{P}x \leqq 1$ for each $x \in X$) if and only if $x_1 \in \Gamma$, $x_2 \in \Gamma$, $x_1 - x_2 \in L^1_\Lambda$ imply $x_1 = x_2$. Moreover, if uniqueness holds for Λ, it holds for $-\Lambda$ and for every translate of Λ as well.

Observe that each positive function x is in Γ if $0 \in \Lambda'$ (take $u=1$).

Suppose $0 \in \Lambda'$, $n \in \Lambda$, $-n \in \Lambda$. Both functions $x_1 = 2$ and $x_2 = 2 - e_n - e_{-n}$ are positive, and their difference is in L^1_Λ. Therefore uniqueness does not hold. The same is true by translation whenever we have $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$, $(\lambda_1 + \lambda_2)/2 \in \Lambda'$. If uniqueness holds and Λ has more than one point, choose two points λ_1, λ_2 in Λ with $|\lambda_2 - \lambda_1|$ minimum. Then $|\lambda_2 - \lambda_1|$ is odd, and it is easy to check that Λ is an arithmetical progression with difference $|\lambda_2 - \lambda_1|$.

Suppose that Λ is finite; using a translation if necessary, suppose $0 \in \Lambda$ and $\Lambda \subset [1-N, N-1]$. Then $u(t) =$ sign sin $2\pi Nt$ belongs to $L^\infty_{\tilde{\Lambda}'}$ (because $\hat{u}(0) = 0$ and $\hat{u}(n) = 0$ if $n \notin N\boldsymbol{Z}$). Since $u \cdot u = |u|$ and $(-u) \cdot (u-1) = |u-1|$, the functions $x_1=u$ and $x_2=u-1$ are in Γ and $x_1-x_2=1$ is in L^1_Λ. Therefore uniqueness in Γ does not hold.

We have proved that a necessary condition for uniqueness is that Λ is an infinite arithmetical progression with an odd ratio. It remains to prove that the metric projection exists in the three typical cases. It will prove convenient to consider $\Lambda=N^+=\{1,2,\cdots\}$ and $\Lambda=(2\nu+1)N^+$ instead of $\Lambda=N$ and $\Lambda=(2\nu+1)N$.

Case $\Lambda=N^+$. Then $L^1_\Lambda=H^1_0$ (functions in H^1 with mean value 0) and $L^\infty_{\Lambda'}=H^\infty$. Given $x\in L^1$, a compactness argument shows that there is at least one $y\in H^1_0$ such that $\|x-y\|=d(x,H^1_0)$ (this uses that F. and M. Riesz theorem: each measure with spectrum in N^+ is absolutely continuous). In order to prove uniqueness, suppose $x\in\Gamma$ and $y\in H^1_0$; then, using u as in (3.2), we have

$$\begin{aligned}\int|x-y| &\geqq \int|u(x-y)| = \int\big||x|-uy\big| \geqq \int\big||x|-\operatorname{Re} uy\big| \\ &\geqq \int(|x|-\operatorname{Re} uy) = \int|x|.\end{aligned} \tag{3.3}$$

If $\|x-y\|=\|x\|$ we have equalities instead of $\geqq$. Since $(1-|u|)x=0$ a.e. the difference between the first integrals is $\int(1-|u|)|y|$, and we have

$$\begin{array}{lll} \text{(i)} & (1-|u|)y=0 & \text{a.e.,} \\ \text{(ii)} & \operatorname{Im} uy=0 & \text{a.e.,} \\ \text{(iii)} & |x|\geqq \operatorname{Re} uy & \text{a.e.,} \end{array} \tag{3.4}$$

In the present case $uy\in H^1_0$, and (ii) gives $uy=0$, therefore (using (i)) $y=0$. This proves uniqueness (Doob's theorem).

Case $\Lambda=(2\nu+1)\boldsymbol{Z}$. For any function $x\in L^1$, let us write

$$x_j(t)=x(t+(j/(2\nu+1))). \tag{3.5}$$

Since $y\in L^1_\Lambda$ means that $y_1=y_2=\cdots=y_{2\nu+1}=y$, $y\in\mathscr{P}x$ means that

$$\int_0^{1/(2\nu+1)}\sum_1^{2\nu+1}|x_j(t)-y(t)|\,dt$$

is as small as possible. We are led to an elementary problem: given $2\nu+1$ points $x_1,\cdots,x_{2\nu+1}$ in the plane (we now write x_j instead of $x_j(t)$), prove that there exists a unique $y\in\boldsymbol{C}$ such that $\sum|x_j-y|$ is minimum. The existence is obvious. To prove uniqueness, suppose $0\in\mathscr{P}x$ and $y\in\mathscr{P}x$. We have by (3.2), $u_jx_j=|x_j|$, $|u_j|\leqq 1$, and $\sum u_j=0$, and (3.4) reads (i) $(1-|u_j|)y=0$ for all j, (ii) $\operatorname{Im} u_jy=0$ for all j; these equalities can be understood for functions a.e., or for complex numbers (t being fixed).

If $y \neq 0$, all u_j have modulus 1 (by (i)) and same argument (by (ii)); since $\sum u_j$ contains an odd number of terms, $\sum u_j \neq 0$. The contradiction gives $y=0$, therefore uniqueness. (There exist more elementary proofs.)

Case $\Lambda=(2\nu+1)N^+$. We keep the notation (3.5). Existence of a best approximation is proved as in the case $\Lambda=N^+$. To prove uniqueness, suppose again $0 \in \mathscr{P}x$ and $y \in \mathscr{P}x$. We have now by (3.2), $u_j x_j=|x_j|$, and $\sum u_j=v \in H^\infty$. By (3.4(ii)), $\operatorname{Im} vy=0$; since $vy \in H_0^1$, this implies $vy=0$, hence $v=0$ or $y=0$. Arguing as before, we get $y=0$.

This ends the proof of (3.1). Using (3.3), it is possible to prove that if $\|x-y\|-\|x\|$ is small, $\|y\|$ is small. This gives

(3.6) *Whenever the metric projection from L^1 to L_Λ^1 exists, it is continuous.*

For more details we refer to [**12**], and also to §5. The analogous situation was investigated by Domar [7] when $L^1(\boldsymbol{R})$ is considered instead of $L^1(\boldsymbol{T})$.

Now we shall proceed to prove (2.19), (2.20) and (2.21).

4. **The case $\Lambda=N^+$. Toeplitz operators.** In this section we write P_l for the linear mapping which takes any formal trigonometric series $\sum_{-\infty}^{\infty} a_n e_n$ into its "Taylor part" $\sum_0^\infty a_n e_n$, and $\bar{P}_l$ for the mapping from $\sum_{-\infty}^{\infty} a_n e_n$ to $\sum_{-\infty}^{0} a_n e_n$. When a trigonometric series is the Fourier series of a function, we identify the series and functions.

Given $x \in L^1$ and y its metric projection on H_0^1, let is write $g=x-y$. Then $g \in \Gamma$ and, by (3.2), there exists $u \in H^\infty$, $\|u\|_\infty \leqq 1$, such that $ug=|g|$ a.e. Taking squares, $u^2g^2=g\bar{g}$; therefore $u^2 g=\bar{g}$. Since $u^2 \in H^\infty$, we have $u^2 y \in H_0^1$; therefore $\bar{P}_l(u^2 g)=\bar{P}_l(u^2 x)$. Everything will depend on the simple formulae

$$\text{(4.1)} \qquad \text{(i) } \bar{P}_l g = \bar{P}_l x, \qquad \text{(ii) } \bar{P}_l \bar{g} = \bar{P}_l(u^2 x).$$

In order to prove (2.19) it suffices to investigate the (linear) operators $x \to \bar{P}_l x$ and $x \to \bar{P}_l(u^2 x)$.

Given $\varphi \in H^\infty$, $\|\varphi\|_\infty \leqq 1$, the Toeplitz operator T_φ is defined as $x \to P_l(\bar{\varphi}x)$. The following theorem will contain both (2.13) and (2.19) (except for the "best condition" part).

(4.2) *The Toeplitz operator T_φ preserves all classes L^r $(1<r<\infty)$, Λ_α^r $(1 \leqq r \leqq \infty,\ 0<\alpha<\infty)$, $C(\{M_n\})$, $\mathscr{F}l^2(\rho)$. It maps Λ_ω^r into $\Lambda_{\omega^*}^r$ when* (2.5) *holds* $(1 \leqq r \leqq \infty)$; *in particular, it maps Λ_ω into C. It maps trigonometric polynomials of order $\leqq N$ into trigonometric polynomials of order $\leqq N$.*

The preservation of $\mathscr{F}l^2(\rho)$ is a theorem of Rabindranathan [**20**].

PROOF OF (4.2). $T_\varphi L^r \subset L^r$ is obvious from $P_l L^r \subset L^r$ (Riesz's theorem on conjugate functions).

Let us prove that T_φ is a contraction of $\mathscr{F}l^2(\rho)$. For each integer ν, let $\mathscr{T}_\nu$ be the set of all $p \in L^2$ such that $\hat{p}(n)=0$ for $n \geqq \nu$ (nothing is assumed for $n<0$). Let $x \in L^2$ and $z=T_\varphi x$;

$$\sum_\nu^\infty |\hat{z}(n)|^2 = \inf_{p\in\mathscr{T}_\nu} \|z - p\|_2^2 = \inf_{p\in\mathscr{T}_\nu} \|\bar{\varphi}x - p\|_2^2 \leqq \inf_{q\in\mathscr{T}_\nu} \|\bar{\varphi}x - \bar{\varphi}q\|_2^2$$

$$\leqq \inf_{q\in\mathscr{T}_\nu} \|x - q\|_2^2 = \sum_\nu^\infty |\hat{x}(n)|^2.$$

Therefore

$$\sum_0^\infty \rho_n |\hat{z}(n)|^2 = \sum_{\nu=0}^\infty (\rho_\nu - \rho_{\nu-1}) \sum_{n=\nu}^\infty |\hat{z}(n)|^2$$

$$\leqq \sum_{\nu=0}^\infty (\rho_\nu - \rho_{\nu-1}) \sum_{n=\nu}^\infty |\hat{x}(n)|^2 = \sum_0^\infty \rho_n |\hat{x}(n)|^2$$

($\rho_{-1}=0$). This is Rabindranathan's inequality.

This inequality together with (2.4) shows that each class $C(\{M_n\})$ is preserved.

Writing $\Delta_n=P_l(V_{n+1}-V_n)$, a look at the Fourier transforms give the equality

$$(4.3)\qquad (V_{n+1} - V_n) * T_\varphi x = \Delta_n * \bar{\varphi}x = \Delta_n * (\bar{\varphi}(x - V_{n-2} * x)).$$

Now $\|\Delta_n\|_1 \leqq 3$, therefore

$$(4.4)\qquad \|(V_{n+1} - V_n) * T_\varphi x\|_r \leqq 3\, \|x - V_{n-2} * x\|_r.$$

If $x \in \Lambda_\alpha^r$ ($1 \leqq r \leqq \infty, 0<\alpha<\infty$), (4.4) together with (2.8) gives $T_\varphi\, x \in \Lambda_\alpha^r$. If $x \in \Lambda_\omega^r$ ($1 \leqq r \leqq \infty$) and ω satisfies (2.5), then (4.4) and (2.7) give $T_\varphi x \in \Lambda_{\omega*}^r$.

The preservation of trigonometric polynomials of order $\leqq N$ is obvious by (4.1(ii)).

This ends the proof of (4.2). In order to have (2.19), we shall have to prove that (2.5) is a best condition for $P\Lambda_\omega \subset C$. Before doing it, let us add a few remarks about (4.2).

(1°) (4.4) together with (2.8) is an easy proof of the theorems $P_l\Lambda_\alpha \subset \Lambda_\alpha$ ($0<\alpha<1$) (Privalov), $P_l\Lambda_* \subset \Lambda_*$ (Zygmund) and $P_l\Lambda_\omega \subset \Lambda_{\omega*}$ when (2.5) holds [**30**, p. 121].

(2°) (4.2) applies to division of analytic functions in the unit disc by inner factors (case $x=P_l x$, $|\varphi|=1$, $x\bar{\varphi}=x\varphi^{-1}=P_l(x\varphi^{-1})$). It proves that division by an inner factor does not affect too much the nice properties of boundary values. This question (preservation of properties by division by inner factors) was studied by V. P. Havin [**10**]; it is also the main purpose of [**20**], [**6**], [**14**], [**16**], [**23**], [**28**].

(3°) The preservation of $C(\{M_n\})$ is almost obvious by looking at Fourier coefficients; it does not need the more refined result on $\mathscr{F}l^2(\rho)$. It includes preservation of analytic functions.

Let us prove now that (2.5) is a best condition for $P\Lambda_\omega \subset C$, P being the metric projection from L^1 to H_0^1.

Suppose (2.5) does not hold. Then there exist two functions

$$f(t) = \sum_0^\infty a_n \cos 2\pi nt, \qquad g(t) = \sum_0^\infty a_n \sin 2\pi nt, \tag{4.5}$$

such that $g \in \Lambda_\omega$, $f \notin L^\infty$, and moreover $f \geqq 0$ [**30**, Chapter 5]. We choose $x = f + y$ and $y = -(f + ig - a_0)$. Then $y \in H_0^1$, $y \notin L^\infty$, $x \in \Lambda_\omega$ and (since $x - y \geqq 0$) $y = Px$.

This ends the proof of (2.19).

As a complement to (2.19), let us state and prove the following theorem (F. Riesz (1920) [**21**]).

(4.6) *Suppose $\hat{x}(n) = 0$ for $n < -N$ and $\hat{x}(-N) \neq 0$ $(N \geqq 1)$. Then the function $g \in L^1$ of minimal norm such that $\hat{g}(n) = \hat{x}(n)$ for $n \leqq 0$ is characterized as follows: $g(t) = G(e^{2\pi it})$ with*

$$G(z) = cz^{-N} \prod_{k=1}^{K} (z - a_k) \prod_{l=1}^{L} (z - b_l) \prod_{m=1}^{M} (z - c_m),$$

$$0 < |a_k| < 1, \quad |b_l| = 1, \quad |c_m| > 1, \quad K + L + M \leqq 2N, \tag{4.7}$$

$$c \prod (-a_k) \prod (-b_l) \prod (-c_m) = \hat{x}(-N),$$

and moreover $M - K > 0$, $M - K = 2\mu$ is even, $a_k = \bar{c}_k^{-1}$ for $1 \leqq k \leqq K$, $c_{K+2j-1} = c_{K+2j}$ for $1 \leqq j \leqq \mu$, $L = 2\lambda$ is even and $b_{2j-1} = b_{2j}$ for $1 \leqq i \leqq \lambda$. In other words, all zeros of $G(z)$ go in pairs, each pair consisting of either two inverse points (like a_k and c_k) or a double point in $|z| \geqq 1$ (like $b_{2j-1} = b_{2j}$, or $c_{K+2j-1} = c_{K+2j}$).

Proof of (4.6). We already know that g is a trigonometric polynomial of order $\leqq N$ (see (4.2) or (4.1(ii))). Since $\hat{g}(-N) = \hat{x}(-N) \neq 0$, we have $g(t) = G(e^{2\pi it})$ with (4.7). Writing also $u(t) = U(e^{2\pi it})$, we can define $U(z)$ as an analytic function in $|z| < 1$ and we have

$$U^2(z) = \frac{\bar{c}}{c} z^{2N-(K+L+M)} \prod \frac{1 - \bar{a}_k z}{z - a_k} \prod \frac{1 - \bar{b}_l z}{z - b_l} \prod \frac{1 - \bar{c}_m z}{z - c_m}$$

(the equality for $|z|=1$ follows from $u^2g=\bar{g}$ and it holds for $|z|<1$ by analytic continuation). Since $U^2(z)$ has no pole in $|z|<1$, each a_k is a $\bar{c}_m^{-1}$; by a reordering of the c_m, $a_k=\bar{c}_k^{-1}$. Since $U^2(z)$ has double zeros in $|z|<1$, the other c_m go in pairs, and by reordering $c_{K+2j-1}=c_{K+2j}$. This already implies that $U(z)G(z)$ is real on $|z|=1$. Since it must be $\geqq 0$, the b_l must go in pairs, say $b_{2j-1}=b_{2j}$. It is quite easy to check that the above conditions allow us to find $u \in H^\infty$ with $ug=|g|$; therefore $g \in \Gamma$.

This fine theorem was originally obtained by variational methods.

5. **The case $\Lambda=(2\nu+1)\boldsymbol{Z}$. Steiner points.** We first consider the following elementary problem. Given n points $x_1, \cdots, x_n$ in $\boldsymbol{C}$, study the set $\{y\}$ of points $y \in \boldsymbol{C}$ such that $\sum_{j=1}^n |x_j-y|$ is minimum.

If $x_1, \cdots, x_n$ are on a line, say $\boldsymbol{R}$, y is also on $\boldsymbol{R}$. Assuming $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, $\{y\}=\{x_{\nu+1}\}$ if $n=2\nu+1$ and $\{y\}=[x_\nu, x_{\nu+1}]$ if $n=2\nu$. This explains the role of odd numbers in our discussion.

From now on, we consider only the case $n=2\nu+1$ (though the case $n=2\nu$ would be interesting too), and $x=(x_1, \cdots, x_n) \in \boldsymbol{C}^n$. If we think of x as an element of $l^1(1, 2, \cdots, n)$, that is we equip $\boldsymbol{C}^n$ with the norm $\|x\|=\sum |x_j|$, we are looking for the metric projection of $X=l^1(1, 2, \cdots, n)$ onto the subspace Y consisting of constants (the diagonal in $\boldsymbol{C}^n$). We use the same letter y for a complex number and for the corresponding constant function, that is $(y, y, \cdots, y)$. When y is the metric projection of x on Y, we write $y=P_0x$. We already know from the discussion of (3.4) that P_0 exists, and we said in (3.6) that it is continuous. We shall prove a more precise result.

(5.1) *The mapping P_0 is lipschitzian of order $\frac{1}{2}$ on every bounded subset of $\boldsymbol{C}^n$.*

PROOF Suppose $P_0x=0$, $P_0x'=y$, $\|x-x'\|<\delta$. We want to bound $\|y\|$. We have

$$\|x-y\| \leqq \|x-x'\| + \|x'-y\| \leqq \|x-x'\| + \|x'\| \leqq \|x\| + 2\delta. \tag{5.2}$$

Our estimate will follow from (5.2).

Thinking of x as a function on $(1, 2, \cdots, n)$, there exists $u=(u_1, \cdots, u_n)$ such that $\sup|u_j|=1$ and $ux=|x|$. Moreover, (3.3) holds, integration meaning now summation on $(1, 2, \cdots, n)$. From (3.3) and (5.2) we obtain

$$\int \big|\,|x|-uy\big| - \int \big|\,|x| - \operatorname{Re} uy\big| \leqq 2\delta.$$

Using the elementary inequality

$$\left(\int |f|\right)^2 \geqq \left(\int |\operatorname{Re} f|\right)^2 + \left(\int |\operatorname{Im} f|\right)^2$$

with $f=|x|-uy$, we get

$$\left(\int |\mathrm{Im}\, uy|\right)^2 \leqq 2\delta\left(\int ||x| - uy| + \int ||x| - \mathrm{Re}\, uy|\right)$$

$$\leqq 4\delta\int |x - y| \leqq 4\delta(\|x\| + 2\delta).$$

If $x\neq 0$, at least one $|u_j|$ is 1, therefore $\|y\|=n|y|\leqq 2n(\delta(\|x\|+2\delta))^{1/2}$. If $x=0$, (5.2) gives directly $\|y\|\leqq 2\delta$. This proves (5.1).

It is natural to ask whether P_0 is actually lipschitzian of order 1, that is $\|P_0x-P_0x'\|\leqq K\|x-x'\|$, K depending only on n. Here is an answer.

(5.2) (i) *P_0 restricted to $\boldsymbol{R}^n$ is lipschitzian of order* 1. (ii) *If $n=3$, P_0 defined on $\boldsymbol{C}^n$ is lipschitzian of order* 1. (iii) *If $n=5, 7, \cdots$, P_0 defined on $\boldsymbol{C}^n$ is not lipschitzian of order* 1 *and is not uniformly continuous.*

PROOF (i) is obvious, with $K=n$.

(ii) If $n=3$, and $y=P_0x$, y is the so-called Steiner point of the triangle (x_1, x_2, x_3). If one angle is $\geqq 2\pi/3$, y is the corresponding summit x_j. Otherwise y is the point such that the half lines $(yx_1)^{\rightarrow}$, $(yx_2)^{\rightarrow}$, $(yx_3)^{\rightarrow}$ make angles $=2\pi/3$. In order to see how y depends on x, let us fix x_1 and x_2 and move x_3 in $\boldsymbol{C}$. Then y moves in a domain bounded by two arcs of circles, and elementary geometry shows that y moves by less than ε if x_3 moves by less than $(\sqrt{3}/2)\varepsilon$.

(iii) If $n\geqq 5$, let us fix $x_1, \cdots, x_{n-1}$ and move x_n in $\boldsymbol{C}$. Then y moves in the domain $|\sum_1^{n-1} (x_j-y)/|x_j-y||^2\leqq 1$. Let us suppose

$$\sum_1^{n-1} \frac{x_j}{|x_j|} = -1 \tag{5.3}$$

and consider the curve L defined by $|\sum_1^{n-1} (x_j-y)/|x_j-y||^2=1$. If $x_n=1$ we have $P_0x=0$ and if we move x_n to $1+i\alpha$ (α small), P_0x (here considered as a complex number) moves on L. We shall show that the angle of L with the real axis at 0 is as small as we want by a convenient choice of $x_1, \cdots, x_{n-1}$. This will prove that P_0 is not lipschitzian of order 1.

Writing $x_j=x_j'+ix_j''$, $y=y'+iy''$,

$$f(y', y'') = \left|\sum_1^{n-1} \frac{x_j - y}{|x_j - y|}\right|^2 = \left(\sum_1^{n-1} \frac{x_j' - y'}{|x_j - y|}\right)^2 + \left(\sum_1^{n-1} \frac{x_j'' - y''}{|x_j - y|}\right)^2,$$

an elementary computation gives

$$\frac{1}{2}\frac{\partial f}{\partial y'}(0, 0) = \sum_1^{n-1} \frac{x_j''^2}{|x_j|^3}, \qquad \frac{1}{2}\frac{\partial f}{\partial y''}(0, 0) = \sum_1^{n-1} \frac{x_j' x_j''}{|x_j|^3}.$$

We choose all x_j outside the domain $x''^2 > (x'^2 + x''^2)^{3/2}$, then

$$|\partial f(0,0)/\partial y'| \leqq 2(n-1).$$

We choose x_{n-1} on the curve γ defined by $x''^2 = (x'^2 + x''^2)^{3/2}$ very near 0, with $x'_{n-1} < 0$, and the other x_j outside the unit disc $|x| \leqq 1$, and such that (5.3) holds (the simplest way to do it is to move a regular polygon in the right position). Since $|x'_{n-1}/x''_{n-1}|$ is very large, $\frac{1}{2}(\partial f(0,0)/\partial y'')$ is very large. That is what we wanted.

We proved actually that P_0 is not lipschitzian of order 1 on any ball $\|x\| < R$. Since P_0 is linear, this means that P_0 is not uniformly continuous. Therefore (5.2) is proved.

A closer examination shows that P_0 is not lipschitzian of order $> \frac{2}{3}$ on any ball $\|x\| < R$. It is not clear whether $\frac{1}{2}$ is the best order in (5.1).

6. **The case $\Lambda = (2\nu+1)N^+$.** We intend to prove (2.21).

Considering $f((2\nu+1)t)$ and $g((2\nu+1)t)$ instead of f and g in (4.5), we see that C and L^∞ are not preserved.

Suppose now $x \in L^r$ $(1 < r < \infty)$ and $y = Px$. Using the notation $x_j(t) = x(t + j/(2\nu+1))$ and the analogue for y and u, we have $y = y_1 = \cdots = y_{2\nu+1} \in H_0^1$, $(x-y)u = |x-y|$, $\|u\|_\infty \leqq 1$, $v = u_1 + \cdots + u_{2\nu+1} \in H^\infty$. Therefore

$$\operatorname{Im} yu_j = \operatorname{Im} x_j u_j \tag{6.1}$$

$$\operatorname{Im} yv = \operatorname{Im} \sum x_j u_j \in L^r, \qquad yv \in H_0^1. \tag{6.2}$$

From (6.2) and the F. and M. Riesz theorem, $yv \in L^r$. Writing E for the set of t such that $|v(t)| \geqq \frac{1}{2}$, we have $y1_E \in L^r$.

On the other hand, there exists $\alpha > 0$ such that, whenever $2\nu+1$ unit vectors make angles smaller than α with a given line, the absolute value of their sum is larger than $\frac{1}{2}$. Therefore, it $t \notin E$, at least one j satisfies $|\operatorname{Im} yu_j| > |y| \sin \alpha$ and, using (6.1), we have also $y(1 - 1_E) \in L^r$. This completes the proof of (2.21).

Bibliography

1. V. M. Adamjan, D. Z. Arov and M. G. Kreĭn, *Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz*, Funkcional. Anal. i Priložen. **2** (1968), no. 1, 1–19. (Russian) MR **38** #2591.

2. R. C. Buck, *Applications of duality in approximation theory*, Approximation of Functions (Proc. Sympos. General Motors Res. Lab., 1964), Elsevier, Amsterdam, 1965, pp. 27–42. MR **33** #4554.

3. L. Carleson, *Projections métriques de L^p sur H^p*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A1159–A1160.

4. L. Carleson and S. Jacobs, *Best uniform approximation by analytic functions*, Ark. Mat. **10** (1972), 219–229.

5. K. deLeeuw and W. Rudin, *Extreme points and extremum problems in* H_1, Pacific J. Math. **8** (1958), 467–485. MR **20** #5426.

6. E. M. Dyn′kin, *Smooth functions on plane sets*, Dokl. Akad. Nauk SSSR **208** (1973), 25–27=Soviet Math. Dokl. **14** (1973), 18–21. MR **47** #2069.

7. Y. Domar, *On the uniqueness of minimal extrapolations*, Ark. Mat. **4** (1960), 19–29. MR **26** #6688.

8. J. L. Doob, *A minimum problem in the theory of analytic functions*, Duke Math. J. **8** (1941), 413–424. MR **3**, 76.

9. P. L. Duren, *Theory of* H^p *spaces*, Academic Press, New York, 1970.

10. V. P. Havin, *The factorization of analytic functions that are smooth up to the boundary*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **22** (1971), 202–205. (Russian) MR **44** #6970.

11. H. Helson, *Note on harmonic functions*, Proc. Amer. Math. Soc. **4** (1953), 686–691. MR **15**, 309.

12. J.-P. Kahane, *Projection métrique de* $L^1(T)$ *sur des sous-espaces fermés invariants par translation*, Proc. Conf. on Approximation (Oberwolfach, 1971), INSM 1972.

13. ———, *Projection métrique dans* $L^1(T)$, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A621–A623.

14. V. E. Katznelson, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **30** (1972), 163–164. (Russian).

15. B. I. Korenbljum, *An extremal property of outer functions*, Mat. Zametki **10** (1971), 53–56=Math. Notes **10** (1971), 456–458. MR **44** #5472.

16. B. I. Korenbljum and V. M. Faivyševskiĭ, *On a class of contracting operators related to the divisibility of analytic functions*, Ukrain. Mat. Ž. **24** (1972), 692–695= Ukrainian Math. J. **24** (1972), 559–561.

17. B. I. Korenbljum and V. S. Korolevič, *Analytic functions which are regular in a disc and smooth on its boundary*, Mat. Zametki **7** (1970), 165–172=Math. Notes **7** (1970), 100–104. MR **42** #4745.

18. A. J. Macintyre and W. W. Rogosinski, *Extremum problems in the theory of analytic functions*, Acta Math. **82** (1950), 275–325. MR **12**, 89.

19. D. J. Newman, *Pseudo-uniform convexity in* H^1, Proc. Amer. Math. Soc. **14** (1963), 676–679. MR **27** #1817.

20. M. Rabindranathan, *Toeplitz operators and division by inner functions*, Indiana Univ. Math. J. **22** (1972/73), 523–529. MR **46** #6085.

21. F. Riesz, *Über Potenzreithen mit vorgeschriebenen Anfangsgliedern*, Acta Math. **42** (1920), 145–171.

22. W. W. Rogosinski and H. S. Shapiro, *On certain extremum problems for analytic functions*, Acta Math. **90** (1953), 287–318. MR **15**, 516.

23. F. A. Šamojan, *Division by an inner function in certain spaces of functions that are analytic in the disc*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **22** (1971), 206–208 (Russian) MR **44** #6973.

24. D. Sarason, *Algebras of functions in the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–299.

25. H. S. Shapiro, Dissertation, M.I.T., Cambridge, Mass., 1952.

26. ———, *Regularity properties of the element of closest approximation*, Trans. Amer. Math. Soc. **181** (1973), 127–142.

27. Ivan Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Die Grundlehren der math. Wissenschaften, Band 171, Springer-Verlag, Berlin and New York, 1970. MR **42** #4937.

28. N. A. Širokov and S. A. Vinogradov, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **22** (1971), 209–211. (Russian)

29. A. Zygmund, *On the preservation of classes of functions*, J. Math. Mech. **8** (1959), 889–895; erratum, ibid. **9** (1960), 663. MR **22** #8277.

30. ———, *Trigonometric series*. Vol. I, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

MATHÉMATIQUES, BATIMENT 425, UNIVERSITÉ PARIS-SUD, 91 ORSAY, FRANCE