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UNIQUENESS OF ORIENTATION PRESERVING PL INVOLUTIONS OF 3-SPACE

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1. **Introduction.** Waldhausen [**2**] has proven that every PL involution of S^3 with 1-dimensional fixed point set is PL equivalent to the one which rotates S^3 around an unknotted simple closed curve. In this note we show how the corresponding result for R^3 (which has been heretofore unknown) follows from a technique used by the second author in his recent paper [**1**]. Specifically, we prove

THEOREM 1. *Every orientation preserving PL involution of R^3 is PL equivalent to the one which rotates R^3 around the z-axis.*

Since such an involution must have 1-dimensional fixed point set, the above theorem is a consequence of the following theorem if one considers the one-point compactification of R^3.

THEOREM 2. *Let h be an involution of a closed 3-manifold M with 1-dimensional fixed point set F. If, for some $x \in F$, there exists a triangulation of $M - \{x\}$ making $h|M - \{x\}$ piecewise linear, then there exists a triangulation of M making h piecewise linear.*

Theorem 2 will be proved by literally imitating the reduction method [**2**, proof of Lemma 2] of Tollefson.

2. **Proof of Theorem 2.**

LEMMA. *Let M, F, x and h be as in Theorem 2. Then, for any neighborhood U of x, there exists in U an invariant 3-cell D containing x in its interior such that $\partial D \cap F \neq \emptyset$ and ∂D is a PL subspace of $M - \{x\}$.*

PROOF. We indicate how to modify the proof of Lemma 2 of [**1**] to produce the desired invariant 3-cell D. We may assume that F is not contained in U. Let Σ be the set of all PL 2-spheres in $M - \{x\}$ that bound 3-cell neighborhoods of x in U and are in h-general position modulo F (in the sense of [**1**]). The lemma follows from the proof of Lemma 2 of [**1**] if the phrase "2-spheres not bounding 3-cells" is replaced by "PL 2-spheres in $M - \{x\}$ bounding 3-cell neighborhoods of x in U."

In order to prove Theorem 2, consider a sequence of invariant 3-cells

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(as in the lemma) $D_1, D_2, \ldots$ such that $D_{i+1} \subset \mathrm{Int}(D_i)$ and $\bigcap D_i = x$. Observe that F meets each 2-sphere ∂D_i in two points. By applying the above-mentioned result of Waldhausen to $h|D_i - \mathrm{Int}(D_{i+1})$ (for each i), we find that $h|D_1$ is topologically equivalent to the cone of $h|\partial D_1$. Now extend the triangulation of $M - \mathrm{Int}(D_1)$ to M by triangulating D_1 as the cone over ∂D_1. This proves Theorem 2.

REFERENCES

1. J. L. Tollefson, *Involutions on* $S^1 \times S^2$ *and other* 3-*manifolds*, Trans. Amer. Math. Soc. (to appear).

2. F. Waldhausen, *Über Involutionen der* 3-*Sphäre*, Topology **8** (1969), 81–91. MR **38** #5209.

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