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DERIVATION RANGES AND THE IDENTITY

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Introduction. If $\mathfrak{A}$ is a C^*-algebra containing the identity and T belongs to $\mathfrak{A}$, then the (inner) derivation induced by T is the operator Δ_T acting on $\mathfrak{A}$ which maps X (in $\mathfrak{A}$) to $TX - XT = [T, X]$. It has been known for many years that I (the identity in $\mathfrak{A}$) is not in the range of Δ_T for any T (I is not a commutator) [4]. J. P. Williams has asked [5] if there is a T in $\mathfrak{A}$ such that I is in the uniform closure of the range of Δ_T. In this paper we show that such T's do exist. In fact, if $\mathscr{A}(\mathfrak{A}) = \{T \in \mathfrak{A} : I$ is in the closure of the range of $\Delta_T\}$ we will show that there is a C^*-algebra $\mathfrak{A}$ such that $\mathscr{A}(\mathfrak{A})$ is uniformly dense in $\mathfrak{A}$. It then follows that $\mathscr{A}(\mathscr{B}(\mathscr{H}))$ is nonempty where $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded linear operators acting on complex infinite dimensional Hilbert space.

Ampliations. In what follows $\mathscr{H}$ will always denote a *separable* infinite dimensional complex Hilbert space. Given T in $B(\mathscr{H})$ the *ampliation* of T is denoted by $T \otimes I$ and is by definition the operator acting on the direct sum of $\aleph_0$ copies of $\mathscr{H}$ which is determined by the matrix

$$T \otimes I = \begin{bmatrix} T & 0 & 0 & \\ 0 & T & 0 & \\ 0 & 0 & T & \\ & & & \cdot \end{bmatrix}.$$

It is well known that the commutant of $T \otimes I$ contains the set of matrices of the form

$$I \otimes S = \begin{bmatrix} s_{11}I & s_{12}I & s_{13}I & \\ s_{21}I & s_{22}I & s_{23}I & \\ s_{31}I & s_{32}I & s_{33}I & \\ & & & \cdot \end{bmatrix},$$

where $S = (s_{ij})$ is the matrix of some element of $\mathscr{B}(\mathscr{H})$. It is also well known that $I \otimes S$ is unitarily equivalent to $S \otimes I$.

In [1] Brown and Pearcy showed that there are sequences of operators X_n and Y_n in $\mathscr{B}(\mathscr{H})$ such that $[X_n, Y_n]$ tends uniformly to the identity as

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$n \to \infty$. It follows easily from the foregoing remarks that $[I \otimes X_n, I \otimes Y_n]$ also tends uniformly to the identity. Furthermore, replacing X_n by $X_n/n\|X_n\|$ and Y_n by $n\|X_n\| Y_n$ if necessary, we may assume that the norm of X_n (and, hence, the norm of $I \otimes X_n$) tends to 0.

Finally since the countable direct sum of separable Hilbert spaces is again a separable Hilbert space, we may view $T \otimes I$ and $I \otimes T$ as elements of $\mathscr{B}(\mathscr{H})$ and, where no confusion can result, this will be done.

The class $\mathscr{E}(\mathfrak{A})$. Let $\mathscr{E}(\mathfrak{A})$ be the set of all T in $\mathfrak{A}$ such that there exist sequences X_n and Y_n in $\mathfrak{A}$ with the properties

(i) $\|X_n\| \to 0$ as $n \to \infty$,

(ii) $\|I - [T + X_n, Y_n]\| \to 0$ as $n \to \infty$.

Note that if $\mathfrak{A}$ is commutative, then $\mathscr{E}(\mathfrak{A})$ is empty. On the other hand, the remarks of the preceding section show that I is in $\mathscr{E}(\mathscr{B}(\mathscr{H}))$. Note also that if T is in $\mathscr{E}(\mathfrak{A})$, then so is UTU^* for all unitary elements U in $\mathfrak{A}$. Most of the remainder of the paper will be devoted to showing that there is a C^*-algebra $\mathfrak{A}$ such that $\mathscr{E}(\mathfrak{A}) = \mathfrak{A}$. Our results then follow easily from the Baire category theorem.

THEOREM 1. *If T is in $\mathscr{B}(\mathscr{H})$ then $T \otimes I$ (viewed as an element of $\mathscr{B}(\mathscr{H})$) is in $\mathscr{E}(B(\mathscr{H}))$.*

PROOF. Let $I \otimes X_n$ and $I \otimes Y_n$ be as in the preceding section. Then, since $T \otimes I$ and $I \otimes Y_n$ commute, $[T \otimes I + I \otimes X_n, I \otimes Y_n] = [I \otimes X_n, I \otimes Y_n]$ which tends uniformly to the identity.

The following facts will be needed in the sequel:

(A) If $\{S_n\}$ is a sequence of operators on a Hilbert space of any dimension, then the C^*-algebra generated by the S_n's (the smallest C^*-algebra containing all the S_n's) is separable.

(B) Let $\mathscr{K}$ be a Hilbert space of dimension 2^c where c denotes the cardinality of the continuum. If T is in $\mathscr{B}(\mathscr{K})$ then $T = \sum \bigoplus T_\alpha$ where each T_α acts on separable Hilbert space. That is, T is the direct sum of elements in $\mathscr{B}(\mathscr{H})$. (If f is a unit vector in $\mathscr{K}$ and $\mathfrak{A}$ is the C^*-algebra generated by T, then $\{Sf : S \in \mathfrak{A}\}$ is a separable subspace of $\mathscr{K}$ which reduces T.)

(C) The cardinality of $\mathscr{B}(\mathscr{H})$ is c (every element of $\mathscr{B}(\mathscr{H})$ is determined by an $\aleph_0$ by $\aleph_0$ matrix).

THEOREM 2. *If T is in $\mathscr{B}(\mathscr{K})$ ($\mathscr{K}$ has dimension 2^c) then T is unitarily equivalent to an operator of the form $T_0 \oplus \sum \bigoplus (T_a \otimes I)$ where the closure of the range of T_0 has dimension less than or equal to c and T_a is in $\mathscr{B}(\mathscr{H})$. Thus, except on a subspace of dimension c, T is unitarily equivalent to a direct sum of ampliations.*

PROOF. From (B) above we know that $T = \sum \bigoplus T_\alpha$ where each T_α

belongs to $\mathscr{B}(\mathscr{H})$. Let $\mathscr{S}_\alpha$ be the (equivalence) class of all indices β such that T_β is unitarily equivalent (in $\mathscr{B}(\mathscr{H})$) to T_α. Then, since there are c operators in $\mathscr{B}(\mathscr{H})$, there are at most c distinct $\mathscr{S}_\alpha$'s. Let $\mathscr{S}_0$ be the union of all those $\mathscr{S}_\alpha$ whose cardinality is less than or equal to c. Then the cardinality of $\mathscr{S}_0$ is less than or equal to c and, thus, the closure of the range of $T_0 = \sum_{\alpha\in\mathscr{S}_0} \bigoplus T_\alpha$ has dimension less than or equal to c. Now consider $T_1 = \sum_{\alpha\in\mathscr{S}_1} \bigoplus T_\alpha$ where $\mathscr{S}_1$ is a class of indices which does not intersect $\mathscr{S}_0$. Then T_1 is the (uncountable) direct sum of unitarily equivalent operators and is, therefore, unitarily equivalent to a direct sum of ampliations. The remainder of the proof is now clear.

The main theorem. Recall that if $\mathscr{I}$ is the uniform closure of the set of all T in $\mathscr{B}(\mathscr{K})$ such that the closure of the range of T has dimension strictly less than 2^c, then $\mathscr{I}$ is a proper two-sided ideal in $\mathscr{B}(\mathscr{K})$ [**2**, Theorem 6.4] and $\mathfrak{A} = \mathscr{B}(\mathscr{K})/\mathscr{I}$ is a C^*-algebra [**3**, p. 43, 1.17.3].

THEOREM 3. *Let* $\mathfrak{A} = \mathscr{B}(\mathscr{K})/\mathscr{I}$. *Then* $\mathscr{E}(\mathfrak{A}) = \mathfrak{A}$.

PROOF. Let T be in $\mathscr{B}(\mathscr{K})$. Then, by Theorem 2, UTU^* has the form $T_0 \oplus \sum\bigoplus(T_a \otimes I)$ for some unitary U in $\mathscr{B}(\mathscr{K})$. Let $X'_n = 0 \oplus \sum\bigoplus(I \otimes X_n)$ and $Y'_n = 0 \oplus \sum\bigoplus(I \otimes Y_n)$ where X_n and Y_n are as in Theorem 1 and the direct sums are formed in the obvious way. Let P be the projection onto the closure of the range of T_0. Then $I - P = \lim_n[UTU^* + X'_n, Y'_n]$. But the dimension of $P\mathscr{K}$ is less than or equal to c. Hence $I - P$ and I belong to the same coset in $\mathfrak{A}$. Thus, the coset containing UTU^* is in $\mathscr{E}(\mathfrak{A})$ and, hence, the coset containing T is in $\mathscr{E}(\mathfrak{A})$.

COROLLARY 4. $\mathscr{A}(\mathfrak{A})$ *is a* (*uniformly dense*) *set of second category in* $\mathfrak{A}$.

PROOF. Let $B_n = \{T \in \mathfrak{A} : \|I - [T, X]\| \geqq 1/n, \text{ for all } X \text{ in } \mathfrak{A}\}$. It is easy to check that B_n is uniformly closed. Now if B_n had nonempty interior for some n then $\mathscr{E}(\mathfrak{A})$ could not be all of $\mathfrak{A}$. Therefore, $\bigcup B_n$ is a set of first category in $\mathfrak{A}$ and $\mathscr{A}(\mathfrak{A}) =$ the complement of $\bigcup B_n$ is a (uniformly dense) set of second category in $\mathfrak{A}$.

COROLLARY 5. $\mathscr{A}(\mathscr{B}(\mathscr{H})) \neq \varnothing$. *That is, there is an operator T in $\mathscr{B}(\mathscr{H})$ such that I is in the uniform closure of the range of* Δ_T.

PROOF. Let $\mathfrak{A}$ be a C^*-algebra such that $\mathscr{A}(\mathfrak{A}) \neq \varnothing$. Represent $\mathfrak{A}$ as an algebra of operators acting on a Hilbert space $\mathscr{K}$ (possibly of high dimension). Let T and Y_n be elements of $\mathfrak{A}$ such that $[T, Y_n]$ tends uniformly to the identity. Then $\mathfrak{A}_1$, the C^*-algebra generated by T and the Y_n's is separable and if f is any unit vector in $\mathscr{K}$, $\{Sf : S \in \mathfrak{A}_1\}$ is a separable subspace of $\mathscr{K}$ which reduces T and each Y_n. Let P be the projection onto this subspace, $T' = PT$ restricted to $P\mathscr{K}$, and $Y'_n = PY_n$ restricted to $P\mathcal{K}$. Then the commutator of T' and Y'_n tends uniformly to the identity operator in $\mathcal{B}(P\mathcal{K})$.

Concluding remarks. By taking direct sums it is clear that $\mathcal{A}(\mathcal{B}(\mathcal{K})) \neq \emptyset$ as long as the dimension of $\mathcal{K}$ is infinite. On the other hand, if $\mathcal{K}$ is finite dimensional an easy (and familiar) trace argument shows that the identity is uniformly bounded away from every commutator.

The class $\mathcal{E}(\mathfrak{A})$ seems to be of interest in its own right. It is easy to show that $\mathcal{E}(\mathfrak{A})$ is uniformly closed and that if T belongs to $\mathcal{E}(\mathfrak{A})$ then so does STS^{-1} for every invertible element S in $\mathfrak{A}$.

In particular, if $\mathcal{C}$ is the Calkin algebra ($\mathcal{B}(\mathcal{H})$ modulo the ideal of compact operators) and T_e denotes the coset in $\mathcal{C}$ which contains the operator T, it can be shown that $\mathcal{E}(\mathcal{C})$ contains every element of the form N_e where N is normal in $\mathcal{B}(\mathcal{H})$ and every element of the form $(Q \oplus 0)_e$ where Q is quasinilpotent and the other direct summand is infinite-dimensional. On the other hand, if V is the unilateral shift of finite multiplicity, we do not know if V_e belongs to $\mathcal{E}(\mathcal{C})$. (Of course, since the shift of infinite multiplicity is an ampliation, its coset must belong to $\mathcal{E}(\mathcal{C})$.)

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