BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

EQUILIBRIUM POSITIONS FOR EQUALLY CHARGED PARTICLES ON A SURFACE[1]

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Communicated by S. S. Chern, November 9, 1972

ABSTRACT. This paper gives a lower bound for the number of equilibrium positions of two or three equally charged particles on an imbedded surface in Euclidean n-space.

Let $f:M \to E^n$ be a C^k $(k \geqq 2)$ imbedding of a closed orientable surface into Euclidean n-space which is generic in a certain sense. This paper announces results on the lower bounds for the number of equilibrium positions of two or three equally charged particles on $f(M)$ and indicates, thereby, the manner in which the general case can be studied. For simplicity all charges are assumed to be $+1$.

1. **The 2 particle case.** The imbedding $f:M \to E^n$ is said to be *V-generic* (*potential-generic*) if the function $V_f: M \times M - D \to \mathrm{R}$ defined on $M \times M$ outside of the diagonal D by

$$V_f(x, y) = 1/\|f(x) - f(y)\|$$

satisfies the property that on $M \times M - D$ all its critical points are nondegenerate. (Any C^k $(k \geqq 2)$ imbedding of M satisfies the property that there exists a real number N such that, if $V_f(x, y) \geqq N$, (x, y) cannot be a critical point of V_f.)

V_f can be easily recognized to be the potential of two unit charges on $f(M)$, so that the critical points of V_f are in fact the equilibrium positions. To compute the lower bound for the number of such positions, one observes that on $M \times M - D$, the critical points of V_f are the same as those of the function V_f^{-2}, that is, the function which assigns to (x, y) the number $\|f(x) - f(y)\|^2$. One may then apply the work of [**1**] to obtain

THEOREM 1. *Let $f:M \to E^n$ be a V-generic imbedding of a surface of genus g into E^n. Then the lower bound for the number of equilibrium positions of two equally charged particles on $f(M)$ is $2g^2 + 3g + 3$.*

2. **The 3 particle case.** The 3 particle case is exceedingly more difficult because of the homology theory involved and thereby gives an indication of the difficulty of the general case.

Consider the triple cartesian product of M with itself, $M \times M \times M$,

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AMS (MOS) subject classifications (1970). Primary 58E05; Secondary 70F10.

Key words and phrases. Equilibrium positions, charged particles, Morse theory, potential, lower bound.

[1] Preparation of this paper was supported in part by NSF Grant No. GP-27576.

and let A be the total diagonal, i.e.

$$A = \{(x, y, z) \in M \times M \times M | x = y \text{ or } x = z \text{ or } y = z\}.$$

The imbedding $f: M \to E^n$ is said to be *V-generic* if the function $V_f: M \times M \times M - A \to R$ defined by

$$V_f(x, y, z) = \frac{1}{\|f(x) - f(y)\|} + \frac{1}{\|f(y) - f(z)\|} + \frac{1}{\|f(z) - f(x)\|}$$

satisfies the property that on $M \times M \times M - A$ all its critical points are nondegenerate. V_f is the potential function for three equally charged particles and its critical points are the equilibrium positions.

It can be shown that there exists a number N such that, if $V_f(x, y, z) \geqq N$, (x, y, z) cannot be a critical point of V_f. Let

$$A_N = \{(x, y, z) \in M \times M \times M | V_f(x, y, z) > N\}.$$

To compute the lower bound for the equilibrium positions, one applies Morse theory to the function V_f on the set $M \times M \times M - A_N$. One finds that the number of critical points of V_f depends on the Betti numbers of the pair $(M \times M \times M, A)$.

To each equilibrium position of V_f, there corresponds six critical points of V_f for if (x, y, z) is a critical point, then so is any triple which is a permutation of x, y, and z. The *index* of an equilibrium position is defined to be the index of the corresponding critical point, so that if c_i is the number of equilibrium positions of index i, V_f has $6c_i$ critical points of index i. The theorem may be stated as follows:

THEOREM 2. *Let b_i be the ith Betti number of $(M \times M \times M, A)$ and let c_i be the number of equilibrium positions of index i. Then*

$$6 \sum_{j=0}^{i} (-1)^j c_{i-j} \geqq \sum_{j=0}^{i} (-1)^j b_{i-j}, \qquad i = 0, \ldots, 6.$$

COROLLARY. *The lower bound for the number of equilibrium positions is*

$$2 \sum_{i=0}^{6} \sum_{j=0}^{i} \left[\frac{(-1)^j b_{i-j}}{6}\right] - \left[\sum_{j=0}^{6} \frac{(-1)^j b_{6-j}}{6}\right],$$

where $[\kappa/6]$ is the smallest integer $\geqq \kappa/6$.

To compute the Betti numbers of $(M \times M \times M, A)$ is rather difficult. The outline of this computation is as follows. First, one easily computes the Betti numbers of $M \times M \times M$. Next one uses Mayer-Vietoris sequences to compute the Betti numbers of A, observing the fact that A is essentially three copies of $M \times M$ joined along a single copy of M. One next calls on the relative exact sequence $\cdots \to H_*(A) \xrightarrow{i_*} H_*(M \times$

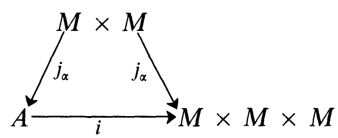
$M \times M) \to H_*(M \times M \times M, A) \to \cdots$ to compute the Betti numbers of $(M \times M \times M, A)$, where i_* is the induced map from the inclusion $i: A \to M \times M \times M$. However, to determine the kernel or image of i_* is by no means an easy task, since this map is not always one-to-one or onto. One proceeds as follows. Let $d: M \to M \times M$ denote the diagonal map, i.e. $d(x) = (x, x)$. Define three maps

$$j_\alpha : M \times M \to M \times M \times M, \qquad \alpha = 1, 2, 3,$$

by

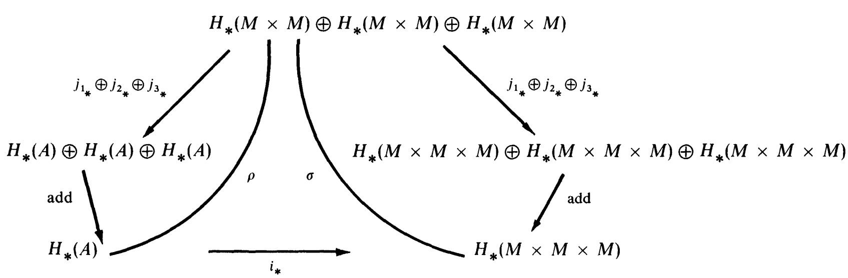
$$j_1(x, y) = (d \times \mathrm{id})(x, y) = (x, x, y)$$
$$j_2(x, y) = (\mathrm{id} \times \mathrm{tw}) \circ (d \times \mathrm{id})(x, y) = (x, y, x)$$
$$j_3(x, y) = (\mathrm{id} \times d)(x, y) = (x, y, y),$$

where id is the identity map, $\mathrm{id}(x) = x$, and tw is the twist map, $\mathrm{tw}(x, y) = (y, x)$. Since the diagrams



$\alpha = 1, 2, 3,$

commute, the following diagram commutes:



where add is just the simple addition in $H_*(A)$ and $H_*(M \times M \times M)$ respectively of the images of j_{1*}, j_{2*}, and j_{3*} in $H_*(A)$ and $H_*(M \times M \times M)$ respectively, and where ρ and σ are the composition maps $\mathrm{add} \circ j_1 \oplus j_{2*} \oplus j_{3*}$.

It is not too difficult a task to determine the kernel and image of ρ; in fact, it is always either an isomorphism or onto. It is, however, quite difficult to determine the image and kernel of σ, but with patience it may be done quite directly since the ring cohomology structure for surfaces is known. Once ρ and σ are completely known one can determine the kernel and image of i_*.

Finally, if one gathers all the information together one obtains that, except for the torus, the Betti numbers of $(M \times M \times M, A)$ are $b_0 = 0$, $b_1 = 0$, $b_2 = 2g^2 + 4g$, $b_3 = 8g^3 + 2g^2 + 2g + 1$, $b_4 = 12g^2$, $b_5 = 6g$, $b_6 = 1$.

Using the corollary we obtain

THEOREM 3. *Let $f: M \to E^n$ be a V-generic imbedding of a surface of genus $g \neq 1$ into E^n. Then the lower bound for the number of equilibrium positions of three equally charged particles on $f(M)$ is*

(a) $(4g^3 + 8g^2 + 6g + 12)/3$ $g \not\equiv 2 \pmod 3$,

(b) $(4g^3 + 8g^2 + 6g + 14)/3$, $g \equiv 2 \pmod 3$.

For the torus special considerations must be made and the lower bound is eleven.

REMARK. The case of three charged particles on a curve in E^n is easily done and the lower bound is found to be two.

The author gratefully acknowledges the valuable assistance of Professor Robert Brown in the homology theory of this work.

REFERENCES

1. F. Takens and J. White, *Morse theory of double normals of immersions*, Indiana J. Math. **21** (1971), 11–17.

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