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GEOMETRY OF LEBESGUE-BOCHNER FUNCTION SPACES—SMOOTHNESS

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ABSTRACT. There exist real Banach spaces E such that the norm in E is of class C^∞ away from zero; however, for any p, $1 \leqq p \leqq \infty$, the norm in the Lebesgue-Bochner function space $L_p(E,\mu)$ is not even twice differentiable away from zero. It is this fact that led to a deeper study of the order of differentiability of the norm function in the spaces $L_p(E, \mu)$, and the main objective of this paper is to announce the complete determination of the order of smoothness of the norm in this class of Banach spaces.

1. **Introduction.** The class of Lebesgue-Bochner function spaces is discussed in Bochner and Taylor [**1**] and has been found to be of considerable importance in various branches of analysis. For their importance in Fourier analysis, the reader is referred to [**1**], and Stein and Weiss [**11**]. Various geometric properties of these spaces have been discussed in Day [**5**], Köthe [**9**], and McShane [**10**]; however, there has been no discussion of higher order smoothness of the norm in this class of Banach spaces. The only known result concerning smoothness is due to McShane [**10**], and his result concerns only the directional derivative (Gâteaux derivative) of the norm. In this paper, a complete characterization of k-times (continuous) differentiability of the norm in $L_p(E, \mu)$ is provided. It might be noted here that the order of smoothness of the Banach spaces $L_p(\mu)$ has been discussed in Sundaresan [**12**], and related results can be found in Bonic and Frampton [**2**].

2. **Definitions and notation.** The definitions and notation used throughout the paper are collected in this section for easy reference. In the following, E denotes a real Banach space. The unit ball of E is $U = \{x \in E \mid \|x\| \leqq 1\}$, and its boundary $S = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E. In what follows, (T, Σ, μ) is an arbitrary measure space, where μ is an extended nonnegative real-valued measure. To avoid trivialities it is assumed that the range of μ contains at least one nonzero real number, and μ is not supported by finitely many atoms.

2.1. DEFINITION. If $1 \leqq p < \infty$, let

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$$\mathscr{L}_p(E, \mu) = \left\{ f \mid f : T \to E \text{ measurable, and } \int_T \|f(t)\|^p \, d\mu(t) < \infty \right\}.$$

Identifying functions in $\mathscr{L}_p(E, \mu)$ which agree μ-a.e., and equipping the resulting linear space with the norm $\|f\| = (\int_T \|f(t)\|^p \, d\mu(t))^{1/p}$, the *Lebesgue-Bochner function spaces* $L_p(E, \mu)$ are obtained.

For an account of these Banach spaces, see Bochner and Taylor [**1**], Dinculeanu [**7**], and Edwards [**8**].

2.2. Definition. Let E and F be Banach spaces, and let $\mathscr{L}(E, F)$ denote the Banach space of *continuous linear mappings* on E into F with the usual operator norm. $\mathscr{B}^k(E, F)$ denotes the Banach space of continuous *k-multilinear mappings* $v : E \times E \times \cdots \times E \to F$ with the norm

$$\|v\| = \sup_{\|x_1\| = \cdots = \|x_k\| = 1} \|v(x_1, \ldots, x_k)\|.$$

The spaces $\mathscr{B}^k(E, F)$ may be identified with the spaces defined inductively as follows:

$$\mathscr{B}^0(E, F) = F, \qquad \mathscr{B}^k(E, F) = \mathscr{L}(E, \mathscr{B}^{k-1}(E, F)) = \mathscr{B}^{k-1}(E, \mathscr{L}(E, F)).$$

2.3. Definition. Let E and F be Banach spaces, and let A be an open subset of E. A mapping $f : A \to F$ is said to be *differentiable at* $x \in A$ if there exists a mapping $f'(x) \in \mathscr{L}(E, F)$ such that

$$\lim_{h \to 0} \|f(x + h) - f(x) - f'(x) \cdot h\| / \|h\| = 0.$$

In this case f is continuous at $x \in A$ and $f'(x)$, which is unique, is called the *derivative of f at x*. The higher order derivatives $f^{(k)} : A \to \mathscr{B}^k(E, F)$ are defined in the usual manner (see Cartan [**3**] or Dieudonné [**6**]). The mapping $f : A \to F$ is said to be of *class C^k* or *k-times continuously differentiable* if it is k-times differentiable and the kth-derivative $f^{(k)} : A \to \mathscr{B}^k(E, F)$ is continuous.

The mapping $f : A \to F$ is said to be of *class C^∞* or *indefinitely continuously differentiable* if it is indefinitely differentiable.

From the results in Cudia [**4**], it follows that if the norm in a Banach space is differentiable away from zero, it is continuously differentiable away from zero.

2.4. Remark. (i) Let E and F be Banach spaces and let A be an open subset of E. If the mapping $f : A \to F$ is k-times differentiable on A, then the k-multilinear mapping $f^{(k)}(x) \in \mathscr{B}^k(E, F)$ is symmetric for each $x \in A$.

(ii) Any continuous k-linear mapping is indefinitely differentiable, and all its derivatives of order $\geqq k + 1$ are zero.

3. **Summary of results.** The order of smoothness of the Lebesgue-Bochner function spaces $L_p(E, \mu)$ is summarized in the following theorems.

3.1. THEOREM. *If* $1 < p < \infty$, *then the norm in* $L_p(E, \mu)$ *is differentiable (hence continuously differentiable) away from zero if and only if the norm in E is differentiable away from zero.*

3.2. THEOREM. *If* $p > k \geqq 2$, *then the norm in* $L_p(E, \mu)$ *is k-times (continuously) differentiable away from zero if and only if the norm in E is k-times (continuously) differentiable away from zero and the kth derivative of the norm in E is uniformly bounded on the unit sphere in E.*

3.3. THEOREM. (i) *If* $p > 2$ *is an odd integer, then the norm in* $L_p(E, \mu)$ *is* $(p-1)$*-times (continuously) differentiable away from zero if and only if the norm in E is* $(p-1)$*-times (continuously) differentiable away from zero and the* $(p-1)$*st derivative of the norm in E is uniformly bounded on the unit sphere in E.*

(ii) *If* $p > 2$ *is not an integer and* $I(p)$ *is the integral part of p, then the norm in* $L_p(E, \mu)$ *is* $I(p)$*-times (continuously) differentiable away from zero if and only if the norm in E is* $I(p)$*-times (continuously) differentiable away from zero and the* $I(p)$*th derivative of the norm in E is uniformly bounded on the unit sphere in E.*

3.4. REMARK. It follows from the results in [**12**], after noting that $L_p(\mu)$ is isometrically isomorphic to a subspace of $L_p(E, \mu)$, that if p is an odd integer, then the norm in $L_p(E, \mu)$ is never p-times differentiable away from zero.

The results are somewhat different when p is an even integer.

3.5. THEOREM. *If* $p \geqq 2$ *is an even integer, then the norm in* $L_p(E, \mu)$ *is p-times (continuously) differentiable away from zero if and only if there exists a continuous homogeneous form T of degree p,* $T: E \times E \times \cdots \times E \to \boldsymbol{R}$, *such that* $\|x\|^p = T(x, x, \ldots, x)$ *for all* $x \in E$.

There is a rather startling corollary to this theorem, namely

3.6. COROLLARY. *The norm in* $L_2(E, \mu)$ *is twice (continuously) differentiable away from zero if and only if E is a Hilbert space.*

Before proceeding to the counterexample mentioned in the abstract, a lemma is needed.

3.7. LEMMA. *If E is a Banach space whose norm is twice differentiable away from zero and the second derivative of the norm is uniformly bounded on the unit sphere in E, then E is uniformly smooth (in particular, E is reflexive).*

This is an immediate consequence of the generalized mean value theorem (see, e.g., Cartan [**3**, p. 70, Theorem 5.6.2]). This lemma together with Theorem 3.2 and Corollary 3.6 imply the following theorem:

3.8. THEOREM. *If $p \geqq 2$ and the norm in $L_p(E, \mu)$ is twice differentiable away from zero, then E is reflexive. Indeed, E is uniformly smooth.*

4. **Example.** In this section an example is discussed, it is this example that led to a deeper study of the order of smoothness of the norm in $L_p(E, \mu)$, $1 \leqq p \leqq \infty$. Before proceeding to the example, a lemma is stated.

4.1. LEMMA [KUIPER]. *The Banach space c_0 is isomorphic to a Banach space whose norm is of class C^∞ away from zero.*

The proof of this lemma can be found on p. 896 of Bonic and Frampton [**2**].

4.2. EXAMPLE. Let E be the Banach space whose norm is of class C^∞ away from zero and which is isomorphic to c_0. Let $1 \leqq p \leqq \infty$, then the norm in $L_p(E, \mu)$ is not even twice differentiable away from zero. The cases $p = 1$ or $p = \infty$ are trivial since in these cases the norm in $L_p(E, \mu)$ is not even once differentiable away from zero. In the case $1 < p < 2$, it follows from the results in [**12**] that the norm in $L_p(E, \mu)$ is not twice differentiable away from zero. Next let $2 \leqq p < \infty$, and suppose the norm in $L_p(E, \mu)$ is twice differentiable away from zero. This implies by Theorem 3.8 that E is reflexive, hence c_0 is reflexive, a contradiction.

The proofs of the preceding theorems will appear elsewhere.

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