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WEIGHTED APPROXIMATION FOR MODULES OF CONTINUOUS FUNCTIONS

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Nachbin [**6**] has enjoyed notable success with the weighted approximation problem in both the real and selfadjoint complex cases, but little progress has yet been realized in the general complex case, even though interest dates from the problem's formative stages (cf. [**4**, p. 1057], [**5**, p. 126]). The purpose of the present note is to provide an answer to this question in a setting which occupies a pivotal position in the theory developed by Nachbin; namely, the bounded case of the weighted approximation problem.

1. **Preliminaries.** In what follows, all functions will be assumed complex valued unless explicitly stated otherwise, while X will denote a completely regular Hausdorff space, $C(X)$ will denote the algebra of all continuous functions on X, and $B_0(X)$ will denote the algebra of all bounded functions on X which also vanish at infinity. We will assume that $B_0(X)$ is equipped with the uniform (convergence) topology induced by $\|\cdot\|$, the usual supremum norm defined on the bounded functions on X.

We now introduce a set V of nonnegative upper semicontinuous functions on X; the elements of V being referred to as *weights*. The corresponding *weighted space* $CV_0(X)$ is the locally convex topological vector space obtained by equipping the linear subspace consisting of those $f \in C(X)$ such that $fv \in B_0(X)$ for every $v \in V$ with the *weighted topology* ω_V generated by the seminorms p_v, one for each $v \in V$, defined on this space by $p_v(f) = \|fv\|$. Since there is no loss of generality in so doing, we will assume that if $u, v \in V$ and $\lambda \geqq 0$, then there is a $w \in V$ for which $\lambda u, \lambda v \leqq w$ (pointwise); i.e., V is a Nachbin family on X [**8**, p. 90]. In addition, we henceforth assume that a subalgebra A of $C(X)$ and a linear subspace W of $CV_0(X)$, where W is an A-module with respect to pointwise multiplication, have been specified. The *weighted approximation problem* [**6**, p. 293] asks for a description of the closure of W in $CV_0(X)$.

2. **The bounded case of the weighted approximation problem.** The setting Nachbin termed the *bounded case* of the weighted approximation problem [**6**, p. 294] is the one in which every $a \in A$ is bounded on the support of each $v \in V$, and it is in this situation that a meaningful characterization of

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$\mathrm{cl}(W)$ can now be given without placing any additional restrictions on A. The description we will give arises from a localization of the weighted approximation problem: denoting the collection of maximal antisymmetric sets for A by $\mathscr{K}_A$, we say that a function $f \in CV_0(X)$ is in $\mathrm{cl}_A(W)$, the *A-local closure* of W in $CV_0(X)$ [**8**, p. 91], if, for any $K \in \mathscr{K}_A$ and $v \in V$, there exists a $w \in W$ such that $\sup\{|f(x) - w(x)|v(x) : x \in K\} \leqq 1$. Clearly, $\mathrm{cl}_A(W)$ is a closed linear subspace of $CV_0(X)$ which also contains W.

2.1. THEOREM. *In the bounded case of the weighted approximation problem, the closure of W in $CV_0(X)$ always coincides with* $\mathrm{cl}_A(W)$.

Before outlining a proof of this theorem, we indicate some of its immediate consequences.

2.2 COROLLARY (NACHBIN [**6**, p. 295]). *If A is selfadjoint, then W is localizable under A in the bounded case of the weighted approximation problem.*

Theorem 2.1 not only contains Nachbin's theorem as a special case, but it can also be regarded as an extension of Bishop's generalized Stone-Weierstrass theorem [**1**] to the class of weighted spaces under discussion. Indeed, as indicated in the earlier paper [**8**] where this theorem was first conjectured, appropriate choices of X and V yield the setting for both Bishop's theorem and Glicksberg's analogous result [**3**] for the strict topology. Moreover, the version of Bishop's theorem recently obtained by Prolla [7, p. 284] and many noteworthy variations of the classical Stone-Weierstrass theorem (see [**8**, p. 97]) are all subsumed by the above theorem.

3. **Proof of Theorem 2.1: an outline.** As did Glicksberg [**3**] and Prolla [**7**] before us, we employ a modification of de Branges' elegant proof of the Stone-Weierstrass theorem [**2**]. However, the key to our argument is in avoiding the need for the characterization of the equicontinuous subsets of $CV_0(X)^*$, the topological dual of $CV_0(X)$, on which the aforementioned proofs so strongly depend.

For each $v \in V$, we let $\mathscr{S}_v = \{gv : g \in CV_0(X)\}$. These linear subspaces of $B_0(X)$ lead to a notion of "support" for elements of $B_0(X)^*$.

3.1. LEMMA. *Assume $T \in B_0(X)^*$. Corresponding to each $v \in V$, there is a set $S_v(T)$ contained in X which satisfies the following properties*:

(1) $S_v(T) \subseteq \mathrm{spt}\, v$;

(2) *if $h \in \mathscr{S}_v$ is such that $h(x) = 0$ for all $x \in S_v(T)$, then $Th = 0$;*

(3) *if there exists $h \in \mathscr{S}_v$ such that $Th \neq 0$, then $S_v(T)$ is not empty;*

(4) *if $T \geqq 0$, then $Th > 0$ whenever $h \in \mathscr{S}_v$, $h \geqq 0$, and $h(x) > 0$ for at least one $x \in S_v(T)$.*

3.2. LEMMA. *Assume T is an extremal point of $(Wv)^0 \cap B_1^0$, where $v \in V$ and $B_1 = \{h \in B_0(X) : \|h\| \leqq 1\}$. In the bounded case of the weighted approximation problem, $S_v(T)$ is an antisymmetric set for A.*

Given the preceding lemmas, de Branges' technique can be applied to obtain the following, and final, step in the proof of Theorem 2.1.

3.3. LEMMA. *In the bounded case of the weighted approximation problem, given $v \in V$, the uniform closure in $B_0(X)$ of Wv always contains* $\mathrm{cl}_A(W)v$.

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