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POSITIVE HARMONIC FUNCTIONS AND BIHARMONIC DEGENERACY[1]

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The class O_{HP} of Riemann surfaces or Riemannian manifolds which do not carry (nonconstant) positive harmonic functions is the smallest harmonically or analytically degenerate class. In particular, it is strictly contained in the classes O_{HB} and O_{HD} of Riemann surfaces or Riemannian manifolds without bounded or Dirichlet finite harmonic functions, and in the classes O_{AB} and O_{AD} of Riemann surfaces without bounded or Dirichlet finite analytic functions.

In the present paper we ask: Are there any relations between O_{HP} and the classes O_{H^2B} and O_{H^2D} of Riemannian manifolds without bounded or Dirichlet finite nonharmonic biharmonic functions? We shall show that the answer is in the negative. Explicitly, if O^N is a null class of N-dimensional manifolds, and $\tilde{O}^N$ its complement, then all four classes

$$O_{HP}^N \cap O_{H^2X}^N, \qquad O_{HP}^N \cap \tilde{O}_{H^2X}^N, \qquad \tilde{O}_{HP}^N \cap O_{H^2X}^N, \qquad \tilde{O}_{HP}^N \cap \tilde{O}_{H^2X}^N$$

are nonempty for both $X = B$ and D, and for any N. This independence of N is of interest, as biharmonic degeneracy often fails to have this property. Typically, whereas the punctured Euclidean N-space is not an element of $O_{H^2B}^N$ for $N = 2, 3$, it does belong to it for all $N \geqq 4$ (Sario-Wang [**6**]).

Methodologically, we introduce in §1 a simple type of Riemannian manifold which, on account of its rectangular coordinates and nonconformal metric, is very versatile in classification problems.

1. We shall show

THEOREM 1. *$O_{HP}^N \cap \tilde{O}_{H^2B}^N \neq \emptyset$ for every N.*

PROOF. Consider the N-manifold, $N \geqq 2$,

$$T = \{0 < x < \infty, 0 \leqq y \leqq 2\pi, 0 \leqq z_i \leqq 2\pi\},$$

$i = 1, \ldots, N-2$, with $y = 0$, $y = 2\pi$ identified, and $z_i = 0$, $z_i = 2\pi$ also identified for every i. Endow T with the metric

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$$ds^2 = dx^2 + x^2 dy^2 + \sum_{i=1}^{N-2} dz_i^2.$$

To see that $T \in O_{HP}$ note that $h(x) \in H(T)$ if $\Delta h = -x^{-1} d(xh')/dx = 0$, that is, $h = a \log x + b$ with constants a, b. Since $|h| \to \infty$ as $x \to 0$ or ∞, the harmonic measure of the ideal boundary of T vanishes, and T belongs to the class O_G of parabolic manifolds. In view of $O_G \subset O_{HP}$ (see e.g. Sario-Nakai [**4**]), we have $T \in O_{HP}$.

An H^2B-function on T is $u = \sin 2y$. In fact,

$$\Delta u = -x^{-1}\,\partial(x^{-1} \cdot 2 \cos 2y)/\partial y = 4x^{-2} \sin 2y$$

and

$$\Delta^2 u = -4x^{-1}\left\{\frac{\partial}{\partial x}[x \cdot (-2x^{-3}) \sin 2y] + \frac{\partial}{\partial y}[x^{-1} \cdot x^{-2} \cdot 2 \cos 2y]\right\} = 0.$$

2. Next we prove

Theorem 2. *$O_{HP}^N \cap O_{H^2B}^N \neq \varnothing$ for every N.*

Proof. Equip the punctured N-space with the metric $ds = r^{-1}|dx|$ so as to obtain a manifold $M = \{0 < r < \infty\}$ with

$$ds^2 = r^{-2} dr^2 + \sum_{i=1}^{N-1} \varphi_i(\theta_1, \ldots, \theta_{N-1})\, d\theta_i^2$$

where the φ_i are trigonometric functions of $(\theta_1, \ldots, \theta_{N-1})$. We have $h(r) \in H(M)$ if $\Delta h = -r^2(h'' + r^{-1}h') = 0$, which gives $h = a \log r + b$. Thus again $M \in O_{HP}$.

To show that $M \in O_{H^2B}$, let S_{nm} be the spherical harmonics, $n = 1, 2, \ldots; m = 1, \ldots, m_n$. For a constant p, a straightforward computation of Δ gives $r^p S_{nm} \in H(M)$ if

$$p = \begin{cases} p_n = \sqrt{n(n + N - 2)}, \\ q_n = -\sqrt{n(n + N - 2)}. \end{cases}$$

Set $h_{nm} = r^{p_n} S_{nm}$, $k_{nm} = r^{q_n} S_{nm}$. The eigenfunction expansion of any $h \in H(M)$ for a fixed r and $\sigma = \log r$ yields

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} h_{nm} + b_{nm} k_{nm}) S_{nm} + a\sigma + b$$

on all of M, with uniform convergence on compact subsets. Again by direct computation, the equations $\Delta u_{nm} = h_{nm}$, $\Delta v_{nm} = k_{nm}$, $\Delta \tau = \sigma$, $\Delta s = 1$ are seen to be satisfied by the functions

$$u_{nm} = -\frac{1}{2p_n} r^{p_n} \log r \cdot S_{nm}, \qquad v_{nm} = -\frac{1}{2q_n} r^{q_n} \log r \cdot S_{nm},$$

$$\tau = -\tfrac{1}{6}(\log r)^3, \qquad s = -\tfrac{1}{2}(\log r)^2.$$

Every $u \in H^2(M)$ has an expansion

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm}u_{nm} + b_{nm}v_{nm}) + a\tau + bs$$
$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm}h_{nm} + d_{nm}k_{nm}) + c\sigma + d$$

on M, with compact convergence implied by that of the expansion of h.

For fixed (n, m),

$$\int_S u \cdot S_{nm}\, d\omega = Ar^{p_n} \log r + Br^{q_n} \log r + Cr^{p_n} + Dr^{q_n},$$

where $d\omega$ is the area element on the unit $(N-1)$-sphere S. If $u \in H^2B(M)$, the integral on the left is bounded in r, and the same is true, by linear independence, of each term on the right. We conclude that $a_{nm} = b_{nm} = c_{nm} = d_{nm} = 0$. The remaining terms in the expansion of u are all radial, and by their linear independence and the boundedness of u we obtain $a = b = c = 0$. Thus u is constant and $M \in O_{H^2B}$.

3. We proceed to show

Theorem 3. *$\tilde{O}^N_{HP} \cap O^N_{H^2B} \neq \emptyset$ for every N.*

Proof. First suppose $N > 2$. Consider the punctured N-space R with the metric $ds = r^{1/3}|dx|$, $r = |x|$. The function $\sigma(r) = r^{-4(N-2)/3}$ is positive and harmonic, hence $R \in \tilde{O}_{HP}$. We now let

$$h_{nm} = r^{p_n}S_{nm}, \qquad k_{nm} = r^{q_n}S_{nm},$$

$$p_n, q_n = \frac{1}{2}\left[-\frac{4}{3}(N-2) \pm \sqrt{\frac{16}{9}(N-2)^2 + 4n(n+N-2)}\right],$$

$$u_{nm} = Ar^{p_n + 8/3}S_{nm}, \qquad v_{nm} = Br^{q_n + 8/3}S_{nm},$$

$$\tau = \begin{cases} C \log r & \text{for } N = 4, \\ Cr^{-4(N-4)/3} & \text{for } N \neq 4, \end{cases}$$

and

$$s = Dr^{8/3}.$$

With this notation, the constants suitably chosen, the reasoning in §2 applies, and we have $R \in O_{H^2B}$.

For $N = 2$, it is known that the disk $|x| < 1$ can be given a conformal metric that excludes H^2B-functions (Nakai-Sario [3]), while harmonicity and hence the existence of HP-functions is not affected.

4. The Euclidean N-ball is trivially in $\tilde{O}^N_{HP} \cap \tilde{O}^N_{H^2B}$ by virtue of $h = r + 1 \in HP$ and $r^2 \in H^2B$. We may therefore summarize our results thus far as follows:

Theorem 4. *The totality of Riemannian N-manifolds decomposes, for every N, into the disjoint nonempty classes*

$$O^N_{HP} \cap O^N_{H^2B}, \qquad O^N_{HP} \cap \tilde{O}^N_{H^2B}, \qquad \tilde{O}^N_{HP} \cap O^N_{H^2B}, \qquad \tilde{O}^N_{HP} \cap \tilde{O}^N_{H^2B}.$$

5. We turn to the relationship of O_{HP} to O_{H^2D}.

Theorem 5. *$O^N_{HP} \cap O^N_{H^2D} \neq \varnothing$ for every N.*

Proof. We recall that the manifold M of §2 is in O_{HP}. To see that it also is in O_{H^2D} we use again the expansion in §2 of $u \in H^2$, which we write as

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{m_n} w_{nm}.$$

Here for $n = 0$,

$$w_{01} = a\tau + bs + c\sigma + d = f_{01},$$

and for $n > 0$,

$$w_{nm} = a_{nm}u_{nm} + b_{nm}v_{nm} + c_{nm}h_{nm} + d_{nm}k_{nm} = f_{nm}S_{nm},$$

with

$$f_{nm} = Ar^{p_n}\log r + Br^{q_n}\log r + Cr^{p_n} + Dr^{q_n}.$$

Choose a fixed (n, m). Then for any $(k, l) \neq (n, m)$ and a fixed $r_0 > 0$, $\Omega = \{x \in M | 0 < r(x) < r_0\}$, the mixed Dirichlet integral over Ω is

$$0 = D_\Omega(h_{nm}, h_{kl}) = \text{const} \int_S \operatorname{grad} S_{nm} \cdot \operatorname{grad} S_{kl}\, d\omega.$$

A fortiori,

$$\begin{aligned} &D_\Omega(w_{nm}, w_{kl}) \\ &\quad= \int_\Omega (\operatorname{grad} f_{nm} \cdot \operatorname{grad} f_{kl}) S_{nm}S_{kl}\, dV + \int_\Omega f_{nm}f_{kl} \operatorname{grad} S_{nm} \cdot \operatorname{grad} S_{kl}\, dV \\ &\quad= 0, \end{aligned}$$

and therefore

$$D(u) \geqq D(w_{nm}) = \text{const} \int_0^\infty f_{nm}\, dr = \infty$$

unless all coefficients (except perhaps d) in the expansion of u vanish.

6. We claim

Theorem 6. *The totality of Riemannian N-manifolds decomposes, for every N, into the disjoint nonempty classes*

$$O_{HP}^N \cap O_{H^2D}^N, \qquad O_{HP}^N \cap \tilde{O}_{H^2D}^N, \qquad \tilde{O}_{HP}^N \cap O_{H^2D}^N, \qquad \tilde{O}_{HP}^N \cap \tilde{O}_{H^2D}^N.$$

Proof. For $N > 2$, the reasoning in §5, with the notation of §3, gives $R \in O_{H^2D}$, hence $\tilde{O}_{HP}^N \cap O_{H^2D}^N \neq \varnothing$. For $N = 2$ this is known (Nakai-Sario [2]).

To see that $O_{HP}^N \cap \tilde{O}_{H^2D}^N \neq \varnothing$, consider the N-ball

$$B_\alpha^N = \{|x| < 1, ds\}, \qquad ds = (1 - |x|^2)^\alpha |dx|.$$

It was proved in Hada-Sario-Wang [1] that

$$B_\alpha^N \in O_G^N \Leftrightarrow \alpha \geqq 1/(N-2) \quad \text{for } N > 2,$$

and

$$B_\alpha^N \in O_{H^2D}^N \Leftrightarrow \begin{cases} \alpha \leqq -\dfrac{3}{N+2} & \text{for } 2 < N \leqq 6, \\ \alpha \notin \left(-\dfrac{3}{N+2}, \dfrac{5}{N-6}\right) & \text{for } N > 6. \end{cases}$$

In particular,

$$B_1^N \in O_{HP}^N \cap \tilde{O}_{H^2D}^N \quad \text{for } 2 < N \leqq 6$$

and

$$B_{1/(N-6)}^N \in O_{HP}^N \cap \tilde{O}_{H^2D}^N \quad \text{for } N > 6.$$

For $N = 2$, the plane can be endowed with a metric which allows H^2D-functions (Nakai-Sario [2] and Sario-Wang [5]), and we have $O_{HP}^2 \cap \tilde{O}_{H^2D}^2 \neq \varnothing$.

The relation $\tilde{O}_{HP}^N \cap \tilde{O}_{H^2D}^N \neq \varnothing$ is again trivial for every N in view of the Euclidean N-ball.

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