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# ZEROS OF SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS

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Let $f(z)$ be a transcendental entire function. If $r_k$ is the radius of the largest disk with center at 0 in which $f^{(k)}(z)$ is zero-free, it is known that, when $f(z)$ is of positive finite order $\rho$ and $\alpha > \rho$, there is an infinite increasing sequence of values of $k$ such that $r_k \geqq k^{(1/\alpha)-1}$ (Ålander [**1**] for $\rho < 1$; stated by Pólya [**4**] for $\rho > 1$ also; the first published proof for $\rho > 1$ was given by Erdös and Rényi [**3**], where Ålander's result is misquoted as being for $\rho > 1$). When $\rho = 1$ and $f(z)$ is of exponential type $\tau$ it is known more precisely that $r_k \geqq c(\tau)$ (Takenaka [**5**]; for modern results see Buckholtz and Frank [**2**]).

We have established the existence of larger zero-free disks if they are no longer required to be centered at 0. Our principal results are as follows.

THEOREM 1. *If $f(z)$ is an entire function at most of order 2, finite type, there is an arbitrarily large disk, somewhere in the plane, in which an infinity of $f^{(k)}(z)$ are zero-free.*

This is a corollary of Ålander's theorem for $\rho < 1$, but not for $1 \leqq \rho \leqq 2$. The conclusion of Theorem 1 fails for entire functions of order greater than 2.

THEOREM 2. *If $\rho > 2$, there is an entire function of order $\rho$ such that, for some positive $A$, every disk, anywhere in the plane, of radius $A$ contains a zero of every $f^{(k)}(z)$.*

THEOREM 3. *If $f(z)$ is an entire function of finite order $\rho \geqq 2$, and $\alpha > \rho$, there is a point $z_0$ such that, for an infinity of $k$, we have $f^{(k)}(z) \neq 0$ in $|z - z_0| < k^{(1/\alpha)-1/2}$.*

Theorem 3 shows that when we do not require the concentric zero-free disks to be centered at a prescribed point, they can be appreciably larger than in Pólya's theorem.

THEOREM 4. *If $f(z)$ is an entire function, for every (arbitrarily large) $c > 0$, there is a $z_0$ such that $f^{(k)}(z) \neq 0$ in $|z - z_0| < ck^{-1/2}$ for an infinity of $k$.*

THEOREM 5. *If $f(z)$ is analytic in $|z| < R$, there are a (possibly small) $c > 0$ and a point $z_0$ in $|z| < R$ such that $f^{(k)}(z) \neq 0$ in $|z - z_0| < ck^{-1/2}$ for an infinity of $k$.*

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Results of this character are not altogether unexpected. For example, if $f(z)$ is of order $\rho$, so is each of its derivatives. Consequently each $f^{(k)}(z)$ has at most $O(R^{\rho+\varepsilon})$ zeros in a disk of radius $R$; when $\rho < 2$ this means that, for each $k$, if $R$ is large enough, $|z| < R$ contains an arbitrarily large disk in which $f^{(k)}(z) \neq 0$. In Theorem 1, however, we have a single disk that is zero-free for each of an infinity of derivatives; and in Theorems 3–5 we have a sequence of concentric disks, of diminishing radii, such that, for a subsequence, each disk is zero-free for the corresponding derivative. To establish such results we require estimates, more precise than those used by Erdös and Rényi, for the number of zeros of $f^{(k)}(z)$ is a disk of prescribed radius.

It is interesting that the dividing line between "small order" and "large order" in this work is at order 2, rather than at order 1 as in the work of Ålander and Pólya; particularly since there are other indications (Pólya [**4**]) that the zeros of successive derivatives tend to become scattered for order less than 2 and to become concentrated for order greater than 2.

## References

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