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THE $\mathcal{S}$-MATRIX ASSOCIATED WITH NONSELFADJOINT DIFFERENTIAL OPERATORS[1]

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1. Let $q(x)$ denote a complex-valued potential defined in R^N, N-dimensional Euclidean space ($N \geqq 1$). Suppose that $q(x)$ satisfies the following condition:

(C) $q(x) \in L_2^{\mathrm{loc}}(R^N)$ and there exist constants $\alpha > 0$, $\rho > 0$, such that

$$\max_{|x|>\rho} |q(x)| e^{\alpha |x|} < \infty.$$

Let A_0 (A) denote the selfadjoint (closed) operator acting in $H = L_2(R^N)$ given by $-\Delta$ ($-\Delta + q(x)\cdot$). It is our intention in this note to study the $\mathcal{S}$-matrix, $\mathcal{S}(\kappa)$ ($\mathcal{S}'(\kappa)$), associated with the operators A_0 and A (A^*).

In two previous papers, [**1**] and [**2**], we derived an abstract scattering theory for two operators A_0 and A acting in a Hilbert space H, where A_0 is selfadjoint and A is closed. We also showed in [**2**] that these results are applicable to the operators A_0 and A defined above.[2] To be more precise, we considered the operators $A_{0_{\mathscr{G}}} = A_0 E_{0_{\mathscr{G}}}$ and $A_{\mathscr{G}} = AE_{\mathscr{G}}$, where $\{E_{0\lambda}\}$ denotes the spectral resolution for the selfadjoint operator A_0, $E_{\mathscr{G}}$ is a projection operator and $\mathscr{G}$ is a closed subinterval of $(0, \infty)$, satisfying the following condition:

($\mathrm{C}_{\mathscr{G}}$) There exists no nontrivial outgoing or incoming solution of the equation $(-\Delta - \lambda + q(x))u(x) = 0$ for any λ in $\mathscr{G}$.[3]

In [**1**] we established the existence of "wave operators," $W^{\pm}$ ($W'^{\pm}$) and the scattering operator, $S = W^{+-1}W^{-}$ ($S' = W'^{+-1}W'^{-}$), associated with $A_{0_{\mathscr{G}}}$ and $A_{\mathscr{G}}$ ($A_{\mathscr{G}}^*$) using a stationary formulation. From this, we obtained the similarity of $A_{0_{\mathscr{G}}}$ and $A_{\mathscr{G}}$ ($A_{\mathscr{G}}^*$). In [**2**] we expressed $W^{\pm}$ ($W'^{\pm}$) in terms of a time-dependent formulation. S was expressed in terms of "distorted plane waves" by means of the "$\mathcal{S}$-matrix" (see §2).

In this paper, we shall obtain a meromorphic continuation of the $\mathcal{S}$-matrix and distorted plane waves from the interval $\mathscr{G}$ to a strip in the

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[2] In [2], the condition on $q(x)$ was weaker than (C). However, we shall need the exponential decay in order to obtain the results of §2.

[3] By outgoing (incoming), we mean that $u(x)$ satisfies the outgoing (incoming) radiation condition: $u(x) = O(|x|^{(1-N)/2})$ and $(\partial/\partial|x| - i\lambda^{1/2})u(x) = o(|x|^{(1-N)/2})$ $((\partial/\partial|x| + i\lambda^{1/2})u(x) = o(|x|^{(1-N)/2}))$ as $|x| \to \infty$.

complex plane. We shall also relate the poles of these functions to "resonant states." The detailed proofs of all of these results will appear elsewhere.

2. Set

$$V_1 = q_1(x) \quad \text{and} \quad V_2 = q_2(x),$$

where

$$q_1(x) = \exp(-\tfrac{1}{2}\alpha|x|) \qquad \text{and} \qquad q_2(x) = q(x)\exp(\tfrac{1}{2}\alpha|x|).$$

Define

$$Q_0^+(\kappa) = V_1 R_0(\kappa^2) V_2 \quad \text{and} \quad P_0^+(\kappa) = V_1 R_0(\kappa^2) V_2^*,$$

for each κ satisfying $\operatorname{Im} \kappa > 0$.

The following result will be proved elsewhere, using the properties of the Green's function for the operator $A_0 - \kappa^2$ and Sobolev's inequality.

LEMMA 1. *Suppose that condition* (C) *holds. Then $Q_0^+(\kappa)$ has a unique continuation to a compact operator acting on $H = L_2(R^N)$ for each κ satisfying* $\operatorname{Im} \kappa > -\frac{1}{2}\alpha$. *Furthermore, $Q_0^+(\kappa)$ is analytic and $Q_0^{+\,-1}(\kappa)$ is meromorphic in κ in the operator topology on H. If in addition* $(C_{\mathscr{G}})$ *holds, then $Q_0^{+\,-1}(\kappa)$ is analytic in a neighborhood of $\mathscr{G}$. Similar results hold for $P_0^+(\kappa)$.*

If we start with $\operatorname{Im} \kappa < 0$, we may obtain compact operators $Q_0^-(\kappa)$ and $P_0^-(\kappa)$ in the same way and they may then be extended to $\operatorname{Im} \kappa < \frac{1}{2}\alpha$ with an analogue of Lemma 1 holding. Now suppose that $|\operatorname{Im} \kappa| < \frac{1}{2}\alpha$ and set $w^0(x; \kappa, \nu) = e^{ix\cdot\kappa\nu}$ and $\tilde{w}^0(x; \kappa, \nu) = q_1(x) w^0(x; \kappa, \nu)$, where $\nu \in S^{N-1}$ (the surface of the unit sphere in R^N). Clearly $\tilde{w}^0(\cdot\,; \kappa, \nu) \in H = L_2(R^N)$.

Set

$$\tilde{w}^\pm(x; \kappa, \nu) = Q_0^{\pm\,-1}(\kappa)(\tilde{w}^0(\cdot\,; \kappa, \nu))(x) \tag{1}$$

and

$$w^\pm(x; \kappa, \nu) = q_1^{-1}(x)\tilde{w}^\pm(x; \kappa, \nu)$$

provided the right side of (1) exists. If $(C_{\mathscr{G}})$ holds and $\kappa = |K|$, where $K = \kappa\nu \in R^N$ and $|K|^2 \in \mathscr{G}$, then $w^0(x; K) = w^0(x; \kappa, \nu)$ is a "plane wave" and $w^\pm(x; K) = w^\pm(x; \kappa, \nu)$ is a "distorted plane wave" (associated with A). We may obtain distorted plane waves, $w'^\pm(x; K)$, associated with A^* similarly with $Q_0^\pm(\kappa)$ replaced by $P_0^\pm(\kappa)$.

We also derived in [2] the following representation for the $\mathscr{S}$-matrix, $\mathscr{S}(\kappa)$ $(\mathscr{S}'(\kappa))$, associated with S (S'), where $\kappa = |K|$, $K = \kappa\nu \in R^N$ and $|K|^2 \in \mathscr{G}$.

$\mathscr{S}(\kappa)$ is a continuous mapping of $L_2(S^{N-1})$ onto itself and is given by

$$S(\kappa)h(\nu) = h(\nu) + \frac{i\kappa}{2\pi}\int_{S^{N-1}} r^+(n, -\nu; \kappa)h(n)\,dn \tag{2}$$

for each $h \in L_2(S^{N-1})$ and $\nu \in S^{N-1}$, where

$$r^+(n, \nu; \kappa) = \int_{R^N} v_2(x; \kappa, n)Q_0^{+\,-1}(\kappa)(v_1(\cdot\,; \kappa, \nu))(x)\,dx$$

and

$$v_j(x; \kappa, n) = q_j(x)w^0(x; \kappa, n),$$

$j = 1, 2$. The operator $\mathscr{S}'(\kappa)$ is defined analogously to $\mathscr{S}(\kappa)$ with $Q_0^+(\kappa)$ replaced by $P_0^+(\kappa)$. Note that while $\mathscr{S}(\kappa)$ is not unitary in general, we do have the relation $\mathscr{S}(\kappa)^* = \mathscr{S}'(\kappa)^{-1}$.

The following result is a simple consequence of Lemma 1.

THEOREM 1. *Suppose that conditions* (C) *and* ($C_{\mathscr{G}}$) *hold. Then the distorted plane waves, $w^\pm(x; \kappa, \nu)$ ($w'^\pm(x; \kappa, \nu)$), and the $\mathscr{S}$-matrix, $\mathscr{S}(\kappa)$ ($\mathscr{S}'(\kappa)$), have meromorphic continuations to $|\text{Im}\,\kappa| < \frac{1}{2}\alpha$ given by* (1) *and* (2), *respectively. The poles occur among those of $Q_0^{\pm\,-1}(\kappa)$ ($P_0^{\pm\,-1}(\kappa)$).*

By a resonant state of A at the point κ_0 in $-\frac{1}{2}\alpha < \text{Im}\,\kappa < 0$ we mean a nontrivial solution, $u(x) \in H^\alpha = L_2\,(R^N; e^{-\alpha|x|}\,dx)$, of the equation

$$\int_{R^N} G_0^+(|x - y|; \kappa_0)q(y)u(y)\,dy = -u(x), \tag{3}$$

where $G_0^+(|x - y|; \kappa_0)$ denotes the outgoing Green's function for the operator $A_0 - \kappa_0^2$. If $q(y)$ is replaced by $\overline{q(y)}$ in (3), we shall say that $u(x)$ is a resonant state of A^* at κ_0. There exists a resonant state of A (A^*) at κ_0 if and only if $Q_0^{+\,-1}(\kappa_0)$ ($P_0^{+\,-1}(\kappa_0)$) fails to exist.

THEOREM 2. *Suppose that condition* (C) *holds. Then A (A^*) has a resonant state at κ_0 if and only if κ_0 is a pole of $\mathscr{S}(\kappa)$ ($\mathscr{S}'(\kappa)$).*

Theorem 2 is proved by obtaining explicit formulas relating the resonant states and the $\mathscr{S}$-matrix. When A is selfadjoint, analogous formulas were obtained by Shenk and Thoe in [3]. In addition, we may show that if $q(x, \varepsilon)$ depends analytically on a complex parameter ε, then so does the $\mathscr{S}$-matrix. Its poles are fractionally analytic functions of ε. Finally we note that all of the above results may be obtained for more general nonselfadjoint operators. The detailed statements and proofs will appear elsewhere.

REFERENCES

1. C. Goldstein, *Perturbation of non-selfadjoint operators*. I, Arch. Rational Mech. Anal. 37 (1970), 268–296.

2. ———, *Perturbation of non-selfadjoint operators*. II, Arch, Rational Mech. Anal. **42** (1971), 380–402.
3. N. Shenk and D. Thoe, *Eigenfunction expansions and scattering theory for perturbations of* $-\Delta$, Lecture Notes, Rocky Mountain Math. Consortium, 1969.

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