BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

THE EXISTENCE OF SOLUTIONS TO CLASSICAL VARIATIONAL PROBLEMS WITHOUT ASSUMING CONVEXITY

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Communicated by Shlomo Sternberg, October 19, 1971

The following is a summary of a paper to appear shortly. It is a part of a Ph.D Thesis submitted to the Senate of the Hebrew University of Jerusalem. It was carried out under the supervision of Professor E. Shamir.

1. We shall deal with the problem of the existence of a solution to the classical variational problem, in parametric form.

2. Let D be a compact region in the x-y plane. (In the following a "compact region" will mean a compact set, such that each two of its points could be joined by a polygon lying entirely (with possible exception of its end-points) in the interior of the set.

3. Let $f(x, y, p, q)$ be defined and have continuous partial derivatives to the second order for every (x, y) in D and every p, q. Also let $f(x, y, p, q)$ be positive homogeneous of the first order in p, q and assume that

$$p^2 + q^2 \neq 0 \Rightarrow f(x, y, p, q) > 0.$$

4. Let P, Q be given points in the x-y plane, and denote by S the family of all curves $(x(t), y(t))$ in D which join P and Q, and such that $x(t)$ and $y(t)$ are absolutely-continuous and satisfy $\dot{x}^2 + \dot{y}^2 \neq 0$ almost everywhere.

5. For each $(x, y) \in D$ we denote by $K(x, y)$ the cone in p, q, f space consisting of the points (p, q, f) for which $f(x, y, p, q) = f$.

6. We shall denote by $\mathrm{In}(x, y)$ the "Indicatrix," that is the set of points in the p-q plane for which $f(x, y, p, q) \equiv 1$.

7. We denote also by $\mathrm{In}^*(x, y)$ the convex-hull of $\mathrm{In}(x, y)$ (in the p-q plane) and we define

$$\mathrm{In}^{**}(x, y) \equiv_{\mathrm{def}} \mathrm{In}(x, y) \cap \mathrm{In}^*(x, y).$$

8. We shall denote by $H(x, y)$ the following set:

$$H(x, y) \equiv_{\mathrm{def}} \left(A \,\middle|\, \sin A = \frac{q}{(p^2 + q^2)^{1/2}}; \cos A = \frac{p}{(p^2 + q^2)^{1/2}}; \quad (p, q) \in \mathrm{In}^{**}(x, y) \right).$$

AMS 1970 *subject classifications*. Primary 49A05; Secondary 28A20.

Key words and phrases. Absolutely continuous function, generalized momenta, conjugate branches, strong variational orientation. Helly principle of choice.

We shall visualize $H(x, y)$ as a subset of angles formed, with the p-axis, by rays (in the p-q plane) which emanate from the origin $p = q = 0$ and cut the indicatrix by points which belong to $\mathrm{In}^{**}(x, y)$.

9. For each "angle" A (not necessarily in $H(x, y)$) and each point $(x, y) \in D$ we shall denote by $D_{(x,y)}(A)$ the ordered pair (of "generalized momenta")

$$D_{(x,y)}(A) \equiv_{\mathrm{def}} (f_p(x, y, \cos A, \sin A), f_q(x, y, \cos A, \sin A));$$

we shall call $D_{(x,y)}(A)$ "the direction of $K(x, y)$ along A."

10. Let $(x, y) \in D$. We shall call A a "branch" of $H(x, y)$ if $A \in H(x, y)$ and if there exists $A_1 \neq A$ such that

$$A_1 \in H(x, y), \qquad D_{(x,y)}(A_1) = D_{(x,y)}(A).$$

It is clear that in this case A_1 is also a branch of $H(x, y)$ and we shall say that "A and A_1 are conjugate to one another in $H(x, y)$." Clearly conjugacy is an equivalence-relation among branches in $H(x, y)$, and we shall call the classes generated by its partition "conjugacy-classes."

11. For each $(x, y) \in D$ and each A_1, A_2 we shall define

$$\begin{aligned} R(x, y, A_1, A_2) \equiv_{\mathrm{def}} & -\cos A_1 \; f_x(x, y, \cos A_2, \sin A_2) \\ & -\sin A_1 \; f_y(x, y, \cos A_2, \sin A_2) \\ & +\cos A_2 \; f_x(x, y, \cos A_1, \sin A_1) \\ & +\sin A_2 \; f_y(x, y, \cos A_1, \sin A_1) \end{aligned}$$

(which is the "Caratheodory function").

We shall also define the following partial ordering:

$$A_1 \prec_{(x,y)} A_2 \Leftrightarrow R(x, y, A_1, A_2) < 0.$$

12. Let $(x, y) \in D$. We shall say that "there exists a strong variational orientation in (x, y)" if the following condition is fulfilled:

Let M be any conjugacy class $\subseteq H(x, y)$; then

(a) $A_1, A_2 \in M \Rightarrow R(x, y, A_1, A_2) \neq 0$,

(b) $(A_1, A_2, A_3 \in M;\ A_1 \prec_{(x,y)} A_2;\ A_2 \prec_{(x,y)} A_3) \Rightarrow (A_1 \prec_{(x,y)} A_3)$.

(That is, $\prec_{(x,y)}$ is an order-relation within each conjugacy-class.)

Furthermore, we shall say that "there exists a strong variational orientation in (x, y)" even in the case that $H(x, y)$ does not contain branches.

13. Let us now assume the following:

(a) The distance between $P \equiv (x_0, y_0)$ and $Q \equiv (x_1, y_1)$ is sufficiently small relative to the distance of either P or Q from the boundary of D.

(b) For each $(x, y) \in D$, each of the conjugacy-classes $\subseteq H(x, y)$ consists of at most a finite number of branches.

(c) There is a strong variational orientation in each $(x, y) \in D$.

REMARK. Assumptions (a), (b) are temporary; assumption (c) replaces "the convexity assumption." It will probably be weakened in the future.

14. THEOREM. *Under the above mentioned assumptions, there exists in S a curve* $(x(t), y(t)) \equiv C$ *for which J attains its absolute minimum, where J is the functional defined by*

$$J(C) \equiv_{\mathrm{def}} \int_{t_0}^{t_1} f(x(t), y(t), \dot{x}(t), \dot{y}(t))\,dt;$$

$$(x(t_0), y(t_0)) \equiv P, \qquad (x(t_1), y(t_1)) \equiv Q$$

(*and the minimum is relative to the "comparison-curves" in S*).

15. The proof is quite complicated. A sketch of it runs as follows:

(a) For each natural number n, construct an n-sided polygon $\mathrm{Pol}^{(n)}$ (joining P and Q) which minimizes J (relative to other n-sided polygons with the same endpoints). It is easily seen that if J_0 is the lower bound of J on S, then

$$\lim_{n\to\infty} J(\mathrm{Pol}^{(n)}) = J_0.$$

(b) Let $(x^{(n)}(s^*), y^{(n)}(s^*))$ be the parametric representation of $\mathrm{Pol}^{(n)}$, where for each n we choose the parameter s^* to be s/L, where s is the length-parameter of $\mathrm{Pol}^{(n)}$ and L is its total length.

(c) Introduce the following notation:

$$R^{(n)}(s^*) \equiv_{\mathrm{def}} \left(\left(\frac{dx^{(n)}(s^*+0)}{ds^*}\right)^2 + \left(\frac{dy^{(n)}(s^*+0)}{ds^*}\right)^2\right)^{1/2},$$

$$\sin A^{(n)}(s^*) \equiv_{\mathrm{def}} \frac{dx^{(n)}(s^*+0)}{ds^*} \Big/ R^{(n)}(s^*),$$

$$\cos A^{(n)}(s^*) \equiv_{\mathrm{def}} \frac{dy^{(n)}(s^*+0)}{ds^*} \Big/ R^{(n)}(s^*),$$

$$f_p^{(n)}(s^*) \equiv f_p(x^{(n)}(s^*), y^{(n)}(s^*), \cos A^{(n)}(s^*), \sin A^{(n)}(s^*)),$$

$$f_q^{(n)}(s^*) \equiv f_q(x^{(n)}(s^*), y^{(n)}(s^*), \cos A^{(n)}(s^*), \sin A^{(n)}(s^*)).$$

Then we prove the following:

(d) $\{f_p^{(n)}(s^*)\}_{(n=1,2,\dots)}$; $\{f_q^{(n)}(s^*)\}_{(n=1,2,\dots)}$ are bounded and have uniformly-bounded variation on $[0 \leqq s^* \leqq 1]$; then, using both the Ascoli and Helly's principles of choice, we prove further

(e) There exists a sequence $\{n_1, n_2, n_3, \dots\}$ such that $\{x^{(n_j)}(s^*)\}_{(j=1,2,\dots)}$ and $\{y^{(n_j)}(s^*)\}_{(j=1,2,\dots)}$ converge uniformly to $x(s^*)$ and $y(s^*)$ respectively, and the "generalized momenta" $\{f_p^{(n_j)}(s^*)\}_{(j=1,2,\dots)}, \{f_q^{(n_j)}(s^*)\}_{(j=1,2,\dots)}$ converge pointwise to $f_p(s^*)$ and $f_q(s^*)$ respectively, where $x(s^*), y(s^*), f_p(s^*)$,

$f_q(s^*)$ are also of B.V., and $x(s^*)$, $y(s^*)$ are continuous with respect to s^*.

Then we prove the following:

(f) For each $s_0^* \in [0, 1]$,

$$\lim_{s^* \to s_0^*;\, j \to \infty} \operatorname{dis}(A^{(n_j)}(s^*), H(x(s_0^*), y(s_0^*))) = 0$$

(where "dis" denotes the natural distance-function).

(g) It follows that as $j \to \infty$ and as $s^* \to s_0^*$, either $A^{(n_j)}(s^*)$ approach a certain limit in $H(x(s_0^*), y(s_0^*))$, or there are finitely-many accumulation points, *which belong to the same conjugacy-class* of $H(x(s_0^*), y(s_0^*))$.

(h) For any $(x_0, y_0) \in D$ and any A_1, A_2, let $R(x_0, y_0, A_1, A_2) < 0$. Then we prove the following:

Let $ABCD$ be a "small" parallelogram in the x-y plane "near" (x_0, y_0), and suppose that AB and CD form an angle A_1 with the x-axis, while BC, DA form an angle A_2 with the x-axis. Then

$$J(AB) + J(BC) < J(AD) + J(DC).$$

(i) By adding the condition of "strong variational orientation" it can be shown, using real-function theory and measure-theoretic considerations, that a subsequence $\{\mathrm{Pol}^{(n_{ji})}\}_{(i=1,2,\dots)}$ could be extracted such that the possibility of more than one accumulation point for $\{A^{(n_{ji})}(s^*)\}_{(i=1,2,\dots)}$ is excluded almost-everywhere.

By using elementary arguments it now follows that $(x(s^*), y(s^*))$ satisfies the conclusions of our theorem.

16. By using this theorem, we show furthermore how one can derive conclusions of the following types:

(a) The existence of solutions to variational problems without assuming 13(a), but using an additional natural assumption concerning the behavior of the problem considered on the boundary of the domain D.

(b) The existence of solutions to variational problems, with possible weakening of condition 12(b).

(c) The existence of solutions to variational problems defined over 2-dimensional manifolds.

No efforts have been made to achieve the most general results. It is hoped that in the near future we shall be able to arrive at theorems that will be more close to necessary and sufficient conditions, and to treat more general types of variational problems.

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