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INVARIANT SUBSPACE THEORY FOR THREE-DIMENSIONAL NILMANIFOLDS

BY LOUIS AUSLANDER[1] AND JONATHAN BREZIN[2,3]

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1. **Introduction.** Let N denote the nilpotent Lie group whose underlying manifold is three-dimensional Euclidean space $\boldsymbol{R}^3$ and whose group operation is given by $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. The subset $\Gamma = \{(a, b, c): a, b, c \in \boldsymbol{Z}\}$ of N is a subgroup, and the quotient N/Γ is a compact manifold, denoted $\boldsymbol{M}$. On the manifold $\boldsymbol{M}$ there is a unique probability measure v invariant under translation by N. (We use right cosets Γg, $g \in N$, and hence translation here means right-translation.) We will use R to denote the regular representation of N on $L^2(\boldsymbol{M}, v)$, namely: $(R_g\phi)(\Gamma h) = \phi(\Gamma hg)$ for all $g, h \in N$ and all $\phi \in L^2(\boldsymbol{M}, v)$.

The representation R decomposes into a direct-sum of irreducible subrepresentations. However, some of the irreducible representations in the sum occur with multiplicity greater than 1, and consequently, $L^2(\boldsymbol{M}, v)$ does not decompose *uniquely* into a direct sum of irreducible R-invariant subspaces. The theorems announced below are aimed toward remedying this situation by introducing into the family of all irreducible R-invariant subspaces of $L^2(\boldsymbol{M}, v)$ a certain amount of structure.

Let $\mathfrak{z}N$ denote the center of N. The Stone-von Neumann theorem says that for each nonzero real number ξ, there is a unique (up to unitary equivalence) irreducible unitary representation U^ξ whose restriction to $\mathfrak{z}N$ is a multiple of the character $(0, 0, z) \mapsto e^{2\pi i\xi z}$ of $\mathfrak{z}N$. We will use $L(\xi)$ to denote the Hilbert space of U^ξ.

It is easy to see that, aside from the characters of N vanishing on Γ, the only irreducible summands of R are those U^ξ where ξ is a nonzero integer. In fact, let n be a nonzero integer, and let $H(n)$ denote the subspace of $L^2(\boldsymbol{M}, v)$ consisting of those functions f satisfying $(R_{(0,0,z)}f)(\Gamma h) = e^{2\pi i n z}f(\Gamma h)$ for all $h \in N$ and $(0, 0, z) \in \mathfrak{z}N$; then the restriction of R to $H(n)$ is unitarily equivalent to the representation $U^n \otimes 1$ of N on $L(n) \otimes \boldsymbol{C}^{|n|}$. (For a proof, see C. C. Moore [**2**].) It follows that the irreducible subspaces of $H(n)$ are in one-to-one correspondence with the space of lines in $\boldsymbol{C}^{|n|}$ through 0—that is, projective space $\boldsymbol{C}P^{|n|-1}$. The theorems below refine this observation.

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[1] John Simon Guggenheim Fellow.
[2] Alfred P. Sloan Fellow.
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2. **Main results.** In accord with the notation already established, we set $H(0)$ equal to the subspace of $L^2(\boldsymbol{M}, v)$ consisting of those functions f constant on orbits of $\mathfrak{z}\boldsymbol{N}$ in $\boldsymbol{M}$. Also, we set $A = H(0) \cap C^\infty(\boldsymbol{M})$. We then have that A is a subalgebra of $C^\infty(\boldsymbol{M})$, and that each $H(n)$ becomes an A-module if we set $(af)(m) = a(m)f(m)$ for all $a \in A, f \in H(n)$, and $m \in \boldsymbol{M}$.

THEOREM 1. *Let n be a nonzero integer, let K be an irreducible R-invariant subspace of $H(n)$, and let $A(K) = \{a \in A : a \cdot f \in K \text{ for all } f \in K\}$. Then $A(K)$ is a subalgebra of A that is closed under complex conjugation, and A is a free $A(K)$-module whose dimension divides n^2 and is divisible by n.*

We define the index of K, ind(K), to be the integer $(\dim_{A(K)} A)/|n|$.

Let $V : H(n) \to L(n) \otimes \boldsymbol{C}^{|n|}$ be an isometric isomorphism that intertwines R and $U^n \otimes 1$. Let $\xi \in \boldsymbol{CP}^{|n|-1}$ and, thinking of ξ as a line in $\boldsymbol{C}^{|n|}$, pick v from among the nonzero points on ξ. Then $V^{-1}(L(n) \otimes v)$ is an irreducible R-invariant subspace $K(\xi)$ depending only on ξ and not on the choice of v.

THEOREM 2. *Let n be a nonzero integer, and let d be a positive integer that divides n. Then*

$$\{\xi \in \boldsymbol{CP}^{|n|-1} : \text{ind}(K(\xi)) \leqq d\}$$

is a nonempty algebraic set of dimension $\leqq d - 1$.

Theorem 2, in particular, says that ind$(K(\xi)) = 1$ for only *finitely many* $\xi \in \boldsymbol{CP}^{|n|-1}$. Our next result characterizes the $K(\xi)$ with index 1. First, a definition:

Let C denote the subgroup $\Gamma\mathfrak{z}\boldsymbol{N}$ of $\boldsymbol{N}$, and for each nonzero integer n, let C_n denote the subgroup $\{(a/n, b/n, z) : a, b \in \boldsymbol{Z}, z \in \boldsymbol{R}\}$ of $\boldsymbol{N}$. Let D be a subgroup of C_n that contains C as a subgroup of index n, and let χ_n denote the character $(a, b, z) \to e^{2\pi i n z}$ of C. It is not hard to see that χ_n can be extended to a character χ_n' of D. The unitary representation of I^D of N induced by χ_n' from D is irreducible by Mackey's little-group theorem (see [**1**]). The representation I^D can be described as follows:

Let μ denote Lebesgue measure on the torus $\boldsymbol{N}/D$, and let $\eta : \boldsymbol{N}/D \to \boldsymbol{N}$ be a section. For all $h \in \boldsymbol{N}$, set $X(h) = \chi_n'(h\eta(h)^{-1})$. Then I^D is given on $L^2(\boldsymbol{N}/D, \mu)$ by $(I_g^D\phi)(Dh) = (X(hg)/X(h))\phi(Dhg)$ for all h, $g \in \boldsymbol{N}$ and $\phi \in L^2(\boldsymbol{N}/D, \mu)$.

The function X is constant on right Γ cosets, and therefore we can map $L^2(\boldsymbol{N}/D, \mu)$ into $H(n)$ by defining $(W^D\phi)(\Gamma h) = X(h)\phi(Dh)$. With W^D so defined, we have $W^D I_g^D = R_g W^D$ for all $g \in \boldsymbol{N}$. Hence the image in $H(n)$ of W^D is an irreducible R-invariant subspace.

We shall say that an irreducible R-invariant subspace K of $H(n)$ is *rationally presentable* if for a suitable choice of D and χ'_n, the subspace K is the image of the map W^D.

THEOREM 3. *Let n be a nonzero integer, and let K be an irreducible R-invariant subspace of $H(n)$. Then the following three conditions on K are equivalent*:

(1) *K is rationally presentable.*

(2) $\operatorname{ind}(K) = 1$.

(3) *There is a function $f \in K$ such that $|f(m)| = 1$ for almost all $m \in \boldsymbol{M}$ and such that $\{a \cdot f : a \in A(K)\}$ is dense in K.*

Making use of the subgroup C_n, we can introduce some structure into the family $Q(n)$ of rationally presentable subspaces of $H(n)$. We begin by observing that if $f \in H(n)$ and if $g \in C_n$, then the correspondence $\Gamma h \mapsto f(\Gamma g^{-1} hg)$ defines a new function, denoted $L_g f$, in $H(n)$. If K and K' are in $Q(n)$, and if $K = L_g K'$ for some $g \in C_n$, we shall call K and K' *inner relatives.*

THEOREM 4. *Inner relatedness is an equivalence relation on $Q(n)$, and each equivalence class contains precisely $|n|$ elements. If $K_1 \in Q(n)$, and if $K_2, \ldots, K_{|n|}$ are the remaining inner relatives of K_1, then $K_1, \ldots, K_{|n|}$ are mutually orthogonal and $H(n) = \sum \bigoplus_{j=1}^{|n|} K_j$.*

For each nonzero integer n, define an epimorphism $\varepsilon_n : N \to N$ by $\varepsilon_n(x, y, z) = (x, ny, nz)$. Then $\varepsilon_n(\Gamma) \subseteq \Gamma$, and thus ε_n induces $\varepsilon_n^* : \boldsymbol{M} \to \boldsymbol{M}$. Let $K_1^{(n)} = \{f \circ \varepsilon_n^* : f \in H(1)\}$. Then $K_1^{(n)} \in Q(n)$. Let $K_2^{(n)}, \ldots, K_{|n|}^{(n)}$ be the inner-relatives of $K_1^{(n)}$. Then $L^2(\boldsymbol{M}, \nu) = H(0) \oplus \sum \bigoplus_{n \neq 0} \sum \bigoplus_{j=1}^{|n|} K_j^{(n)}$. Using families of epimorphisms other than the family $\{\varepsilon_n\}$, we can generate other direct-sum decompositions of $L^2(\boldsymbol{M}, \nu)$. One corollary of all of this is the following theorem:

THEOREM 5. *Let f be a real-analytic function on $\boldsymbol{M}$, and let K be any irreducible R-invariant subspace of $L^2(\boldsymbol{M}, \nu)$. Then the orthogonal projection of f onto K is also real-analytic.*

Indeed, if K is in $H(0)$, or is $H(1)$, Theorem 5 is obvious; the theorem follows in general by working with the spaces $K_j^{(n)}$.

We remark, in conclusion, that all of our results generalize without difficulty to 2-step nilpotent Lie groups in general. For more complicated nilpotent Lie groups, the situation at present is not very clear, and is being worked on.

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DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10036

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address: (L. AUSLANDER) INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540