BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

COHERENCE FOR CATEGORIES WITH ASSOCIATIVITY, COMMUTATIVITY AND DISTRIBUTIVITY

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Communicated by Saunders Mac Lane, August 5, 1971

Let $\mathscr{C}$ be a category, $\otimes, \oplus : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ two functors, U and N objects of $\mathscr{C}$. Suppose that for any objects, A, B, C of $\mathscr{C}$ we have natural isomorphisms,

$$\begin{array}{lll} & \multicolumn{2}{c}{\alpha_{A,B,C}: A \otimes (B \otimes C) \to (A \otimes B) \otimes C,} \\ & \multicolumn{2}{c}{\alpha'_{A,B,C}: A \oplus (B \oplus C) \to (A \oplus B) \oplus C,} \\ & \lambda_A: U \otimes A \to A, & \lambda'_A: N \oplus A \to A, \\ \text{(I)} & \rho_A: A \otimes U \to A, & \rho'_A: A \oplus N \to A, \\ & \gamma_{A,B}: A \otimes B \to B \otimes A, & \gamma'_{A,B}: A \oplus B \to B \oplus A, \\ & \multicolumn{2}{c}{\lambda^*_A: N \otimes A \to N,} \\ & \multicolumn{2}{c}{\rho^*_A: A \otimes N \to N,} \end{array}$$

and natural monomorphisms

$$\text{(II)} \qquad \begin{array}{l} \delta_{A,B,C}: A \otimes (B \oplus C) \to A \otimes B \oplus A \otimes C, \\ \delta'_{A,B,C}: (A \oplus B) \otimes C \to A \otimes C \oplus B \otimes C. \end{array}$$

Roughly speaking the coherence problem is to determine the conditions (denoted coherence conditions) in which the arrows obtained by combining elements of type (I), (II) and identities with $\otimes$ and $\oplus$ only depend on the domain and codomain of the arrow. This note is to announce an answer to this question that was proposed in [**6**] as raised by H. Bass.

The first coherence results stated as such are contained in [**5**] which treats the case of only one functor $\otimes$. The solution for a more complicated situation in closed categories is given in [**4**]. Other papers with results related to coherence problems are listed in the references.

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Take a set, $X = \{x_1, x_2, \ldots, x_p, n, u\}$, and construct $\mathscr{A}$, the free $\{\cdot, +\}$-algebra over it. Let $\mathscr{G}$ be the graph of all formal symbols, for $x, y, z \in \mathscr{A}$,

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AMS 1970 *subject classifications*. Primary 18D99; Secondary 18D10.

Key words and phrases. Coherent, coherence conditions, distributivity, regular element, graph.

$$\alpha_{x,y,z}: x(yz) \to (xy)z, \qquad \alpha'_{x,y,z}: x + (y + z) \to (x + y) + z$$
$$\lambda_x: ux \to x, \qquad \lambda'_x: n + x \to x,$$
$$\rho_x: xu \to x, \qquad \rho'_x: x + n \to x,$$
$$\gamma_{x,y}: xy \to yx, \qquad \gamma'_{x,y}: x + y \to y + x,$$
$$\lambda^*_x: nx \to n,$$
$$\rho^*_x: xn \to n,$$

their formal inverses and

$$\delta_{x,y,z}: x(y + z) \to xy + xz,$$
$$\delta'_{x,y,z}: (x + y)z \to xz + yz,$$
$$1_x: x \to x.$$

Construct $\mathscr{H}$, the free $\{\cdot, +\}$-algebra over $\mathscr{G}$ and take on $\mathscr{H}$ the only extension of the graph structure of $\mathscr{G}$ in which the projections are $\{\cdot, +\}$-morphisms. One element of $\mathscr{H}$ is said to be an *instantiation* if, with at most one exception, only elements of $\mathscr{G}$ of type 1_x occur in its expression. We will denote by $\mathscr{I}$ the graph of all the instantiations of $\mathscr{G}$.

Fix now p objects, $C_1, C_2, \ldots, C_p$, of $\mathscr{C}$ and let $g: \mathscr{I} \to \mathscr{C}$ be the morphism of graphs defined on the vertices by the conditions (i) $gu = U$, $gn = N, gx_i = C_i$, for $1 \leqq i \leqq p$, (ii) $g(x + y) = gx \oplus gy, g(xy) = gx \otimes gy$, for $x, y \in \mathscr{A}$, on $\mathscr{G}$ by taking each formal symbol onto the arrow of $\mathscr{C}$ determined replacing each subscript by its image by g and such that for $x, y \in \mathscr{I}$, $g(x + y) = gx \oplus gy$, $g(xy) = gx \otimes gy$. This definition depends upon the C_i and allows us to define the value of a path with steps in $\mathscr{I}$ as the product of the images of the steps.

Let $\mathscr{A}^*$ be the free $\{\cdot, +\}$-algebra over X, with associativity and commutativity for $\cdot$ and $+$, distributivity of $\cdot$ relatively to $+$, null element n, identity element u, and the additional condition $na = n$ for $a \in \mathscr{A}^*$. The identity map of X defines a $\{\cdot, +\}$-morphism $f: \mathscr{A} \to \mathscr{A}^*$. An element, a, of $\mathscr{A}$ is defined to be *regular* if fa can be expressed as a sum of different elements of $\mathscr{A}^*$, each of which is a product of different elements of X.

Our coherence conditions require the commutativity of the diagrams of a finite family of types. Roughly speaking our conditions are equivalent to the commutativity of any diagram that can be constructed taking, for each vertex, the iteration by $\otimes$ and $\oplus$ of not more than four objects, equal or different, of $\mathscr{C}$ and such that each edge is an iteration by $\otimes$ and $\oplus$ of arrows of type (I), (II) and identities: we are reduced to a finite number of types of diagrams if we drop unnecessary commutativity conditions.

With the above definitions we can state the following theorem which is our main result.

COHERENCE THEOREM. *If $\mathscr{C}$ satisfies the coherence conditions and a is regular then the value of any path from a to b, whose steps are in $\mathscr{I}$, depends only upon a and b.*

A detailed exposition of these results will appear elsewhere.

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