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TAMING IRREGULAR SETS OF HOMEOMORPHISMS

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1. **Introduction.** Let $\mathscr{U}$ be an n-dimensional open connected manifold, $\mathscr{U}^\infty = \mathscr{U} \cup \{\infty\}$ the one-point compactification of $\mathscr{U}$, and d a metric on $\mathscr{U}^\infty$. Suppose that h is a homeomorphism of $\mathscr{U}$ onto itself and let h_∞ be the extension of h to $\mathscr{U}^\infty$. If $p \in \mathscr{U}^\infty$, we say that h is *regular at* p if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(p, q) < \delta$ implies that $d(h^n_\infty(p), h^n_\infty(q)) < \varepsilon$ for all n. If h is not regular at p, we say that p is an *irregular point* of h.

Homeomorphisms with finitely or countably many irregular points have been studied extensively [**4**]-[**10**], [**12**]. In this paper, we consider homeomorphisms h which satisfy

(1) the set of irregular points of h is $P \cup \{\infty\}$, where P is a k-dimensional continuum with $k \leqq n - 2$,

and seek conditions on h which imply that P is nicely embedded. Details of proofs will appear elsewhere.

2. **Nice homeomorphisms.** Suppose that $\mathscr{U}$ and h are as above. We say that h is a *nice homeomorphism* if h satisfies (1),

(2) for each $x \in \mathscr{U} - P$, $\overline{\lim}_{n\to\infty} h^n(x) \subset P$ and $\overline{\lim}_{n\to-\infty} h^n(x) = \infty$, and

(3) the mapping $f_h : \mathscr{U} \to P$ given by $f_h(x) = \lim_{n\to\infty} h^n(x)$ exists and is continuous.

REMARKS. If h satisfies (1), the work of T. Homma and S. Kinoshita [**5**] can be used to show that either h or h^{-1} satisfies (2), so that the strength of our assumptions is in (3). For example, let $h: S^1 \times R^2 \to S^1 \times R^2$ be defined by $h(x, t) = (k(x), \frac{1}{2}t)$ where $k: S^1 \to S^1$ is rotation through an irrational multiple of π radians. Then h satisfies (1) and (2) with $P = S^1 \times \{0\}$, but h does not satisfy (3).

The canonical example of a nice homeomorphism is the case where $\mathscr{U}$ is an open mapping cylinder over P and h is a homeomorphism which "pushes in" along the product structure.

PROPOSITION 1. *If h is a nice homeomorphism, then*
(i) *P is an absolute neighborhood retract;*
(ii) *f_h is onto;*
(iii) *the fixed point set of h is P;*

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(iv) *the inclusion* $P \subseteq \mathscr{U}$ *is a homotopy equivalence*;

(v) *the natural projection* p *of* $\mathscr{U} - P$ *onto the orbit space* $\hat{\mathscr{U}}$ *of* $h|\mathscr{U} - P$ *is a covering map*;

(vi) $\hat{\mathscr{U}}$ *is a closed n-manifold*; *and*

(vii) f_h *induces a map* $\hat{f}_h : \hat{\mathscr{U}} \to P$ *such that* $\hat{f}_h p = f_h$.

(i)–(iv) follow from point set arguments and the fact that $hf_h = f_h$. (v)–(vii) follow from elementary facts about covering spaces and [**11**].

3. **AFG sets and maps.** If X is a continuum in the ENR M, we say that X has *property AFG* if there is a neighborhood W of X in M such that for each neighborhood U of X in W there is a neighborhood V of X, $V \subset U$ such that each map of S^1 into V which is null homologous in U is null homotopic in U.

It can be shown, in the spirit of [**13**], that the AFG property depends only on the homotopy type of X.

If f is a proper map between manifolds, we say that f is an *AFG map* provided that $f^{-1}(x)$ has property AFG for each x in the image of f.

4. **Taming irregular sets in high dimensions.** If P is a polyhedron in $\mathscr{U}$, we say that P is *locally flat* if P has a triangulation in which each simplex is locally flat.

THEOREM 2. *If* h *is a nice homeomorphism with* P *a polyhedron*, $n \geqq 6$, *and* $k + 3 \leqq n$, *then* P *is locally flat if and only if* $\hat{f}_h$ *is an AFG map.*

Theorem 2 is proven by using the homotopy properties of $\hat{f}_h$ to show that P is locally nice and by applying Bryant and Seebeck [**3**]. An important step in the proof is the application of L. Siebenmann's obstruction theory [**15**] to prove

THEOREM 3. *If* $\hat{f}_h$ *is AFG and* B *is the open star of some point in* P *in some triangulation of* P, *then* $\hat{f}_h^{-1}(B)$ *is homeomorphic to the interior of a compact manifold provided* $n \geqq 6$.

5. **The three-dimensional case.** If h is a nice homeomorphism, we say that h has a *cross-section* if there is a closed, locally flat $(n-1)$-manifold $T \subset \mathscr{U} - P$ such that $f_h^{-1}(x) \cap T$ is a continuum for each $x \in P$, T separates $\mathscr{U}$ into two components with P in the bounded component, and $h(T) \cap T = \varnothing$.

THEOREM 4. *Let* h *be a nice homeomorphism with cross-section*, $n = 3$, *and* $k = 1$. *Then* P *is locally tame at each point and* $\mathscr{U}$ *is homeomorphic to the interior of a cube with* q *handles, where* $q = \operatorname{rank} H_1(P)$.

The proof of Theorem 4 is a lengthy argument using standard tools in three-dimensional topology. An important step in the proof involves an appeal to a taming theorem of D. R. McMillan [**14**].

If $p \in \mathscr{U}$, we say that h is *positively regular at* p if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(p, q) < \delta$ implies $d(h^n(p), h^n(q)) < \varepsilon$ for all $n > 0$.

PROPOSITION 5. *If h satisfies* (1) *and* (2), $k = 1$, $P \not\cong S^1$, $h|P =$ *identity, and h is positively regular on* $\mathscr{U}$, *then h is a nice homeomorphism.*

Theorem 4, then, has an obvious restatement in terms of positive regularity. Examples can be given to show that Theorem 4 cannot be extended to higher dimensions. In fact, the construction of M. Brown [**2**] using the Andrews-Curtis Theorem [**1**] can be used to construct, for each $n \geqq 4$ and $1 \leqq k \leqq n - 3$, a homeomorphism h which satisfies (1) and (2) with $\mathscr{U} = \boldsymbol{R}^n$ and P a wildly embedded k-cell, such that h has a cross-section and is positively regular on $\boldsymbol{R}^n$.

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