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VARIATIONAL PROBLEMS WITHIN THE CLASS OF SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

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1. **Introduction.** A classical problem in the calculus of variations is the optimization of a multiple integral over a domain G of a function containing as arguments the independent variables, the unknown function and its partial derivatives up to order l. Usually the unknown function is required to be an element of the class of all functions that are $2l$-times continuously differentiable defined on an open domain containing G.

The optimization problem that is dealt with in this paper differs from the one above in that the class of admissible functions to be considered is the collection of all sufficiently smooth solutions in G of a given partial differential equation of order greater than or equal to $2l$.

This paper contains the definition of the variational adjoint, a special form of the variational adjoint boundary conditions, and necessary conditions for the elliptic as well as for the parabolic case. The necessary conditions take the form of a boundary value problem.

A physical application occurs in the control with boundary and initial conditions of a process in G that is described by a specific partial differential equation. If the differential equation is of elliptic type the process may be controlled by Dirichlet boundary conditions or by any other set of boundary conditions that generate a class of admissible functions.

2. **Notation and definitions.** R^ν is the ν-dimensional Euclidean space. G is an open bounded domain in R^ν with boundary ∂G. $\partial G \in C^k$ denotes that ∂G is k-times continuously differentiable. If $\partial G \in C^1$ then $\boldsymbol{n}$ is the outward unit normal vector to ∂G. If $A \subset R^\nu$, then nbh A is an open set in R^ν that contains A. If $u \in C^k(G)$, then $\|u(x)\|_k$ denotes the sum of the supremums in G of the absolute values of the function u and all its derivatives of order less or equal to k. If $\alpha \in [1, \nu]$ (α an integer) then $D_\alpha u(x)$ denotes $\partial u(x)/\partial x_\alpha$, $u_{,\alpha}$ is the same as $D_\alpha u(x)$. If $\partial G \in C^k$ and if $u \in C^k(\text{nbh } \partial G)$ then $(\partial/\partial \boldsymbol{n})^k u(x)$ is the kth derivative of u along $\boldsymbol{n}$ on ∂G.

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Convention. Summation is taken over indices α_i if they are repeated in the same term. The summation indices α_i always run independently from 1 to ν. If the first index in a sequence of indices has a serial number that is one bigger than that of the last index in the subscript (or superscript), then the quantity is interpreted as one without any subscript (or superscript) (e.g. if $j=2$ then $u_{,\alpha_3,-,\alpha_j}$ should be read as u).

The *differential expression* L is given by

$$(2.1) \qquad L = \sum_{k=0}^{m} \overset{\alpha_1,-,\alpha_k}{A(x)} D_{\alpha_1} \cdots D_{\alpha_k}, \qquad x \in G.$$

The coefficients $\overset{\alpha_1,-,\alpha_k}{A(x)}$ are assumed to be symmetric in the indices $\alpha_1,$—$,\alpha_k$. L^* is the formal adjoint of L.

The *functional* J, defined on $C^l(\overline{G})$, is given by

$$(2.2) \qquad J(u) = \int_G F(x, u(x), Du(x),—,D^l u(x))\, dV.$$

Here it is assumed that: $l \in [1, m]$; $F \in C^2(R^\eta)$, $\eta = \sum_{k=0}^{l} (\nu^k)$; and $\partial F/\partial u_{,\alpha_1,-,\alpha_j}$ is symmetric in $\alpha_1,$—$,\alpha_j$ for $2 \leqq j \leqq l$.

For a given function $f \in C(G)$, the function spaces U, U_H, $U^{(i)}$ and $U_H^{(i)}$ are defined by

(2.3) $u \in U$ iff (1°) $L(u(x)) = f(x)$, $x \in G$, (2°) $u \in C^m(G)$, (3°) $J(u)$ is defined.

(2.4) $u \in U_H$ iff (1°) $L(u(x)) = 0$, $x \in G$, (2°) $u \in C^m(G)$.

(2.5) $U^{(i)} = U \cap C^i(\overline{G})$; $U_H^{(i)} = U_H \cap C^i(\overline{G})$.

The *class of admissible functions* W and the corresponding *class of admissible variations* W_H are chosen in accordance with

(2.6) (1°) W is a nonempty subset of U; (2°) $W_H = \{\delta u : \delta u = u_1 - u_2 \wedge u_1 \in W \wedge u_2 \in W\}$; (3°) W is chosen in such a way that W_H is a linear function space.

DEFINITION. J has a relative extremum within W at $u \in W$ if there is a positive real number δ, such that $J(u+\delta u) - J(u)$ is either definite positive or definite negative for all $\delta u \in W_H$ that satisfy: $\|\delta u\|_l$ exists and is smaller than δ.

3. The variational adjoint.

THEOREM 1. *Assume that*: (1°) $F \in C^{l+1}(R^\eta)$; $u \in C^{2l}(\overline{G})$; $\delta u \in U_H^{(m)}$; $\partial G \in C^m$. (2°) $A^{\alpha_1,-,\alpha_k} \in C^k(\overline{G})$, *for* $0 \leqq k \leqq m$. (3°) *There exists a function* $v \in C^m(\overline{G})$ *that satisfies* $L^*(v(x)) = [F]_{,u}$, *for* $x \in G$. *Then* $\delta J(u; \delta u)$, *the principal linear part of the variation, can be written as*

$$\int_{\partial G}\left\{\sum_{j=0}^{l-1}\delta u_{,\alpha_1,\cdots,\alpha_j}\left\{Q^{\alpha_1,\cdots,\alpha_j}-\sum_{i=0}^{m-j-1}v_{,\alpha_{j+1},\cdots,\alpha_{j+i}}\overset{\alpha_1,\cdots,\alpha_{i+j}}{P(i;j)}\right\}\right.$$
$$(3.1)\qquad \left.-\sum_{j=l}^{m-1}\delta u_{,\alpha_1,\cdots,\alpha_j}\sum_{i=0}^{m-j-1}v_{,\alpha_{j+1},\cdots,\alpha_{j+i}}\overset{\alpha_1,\cdots,\alpha_{i+j}}{P(i;j)}\right\}dS.$$

Here $[F]_{,u}$, *the Euler expression, is given by*

$$(3.2)\qquad [F]_{,u}=\sum_{k=0}^{l}(-1)^kD_{\alpha_1}\cdots D_{\alpha_k}\left(\frac{\partial F}{\partial u_{,\alpha_1,\cdots,\alpha_k}}\right),\qquad x\in G.$$

The expressions $Q^{\alpha_1,\cdots,\alpha_j}$, $0\leqq j\leqq l-1$, $x\in\partial G$, *are given by*

$$(3.3)\qquad Q^{\alpha_1,\cdots,\alpha_j}=\sum_{k=j+1}^{l}(-1)^{k-j-1}\left\{D_{\alpha_{j+1}}\cdots D_{\alpha_{k-1}}\left(\frac{\partial F}{\partial u_{,\alpha_1,\cdots,\alpha_k}}\right)\right\}n_{\alpha_k}.$$

The functions

$$\overset{\alpha_1,\cdots,\alpha_{i+j}}{P(i;j)},\quad 1\leqq i\leqq m-1,\ 1\leqq j\leqq m-i-1,\ x\in\partial G,$$

are given by

$$(3.4)\qquad \overset{\alpha_1,\cdots,\alpha_{i+j}}{P(i;j)}=\sum_{k=i+j+1}^{m}(-1)^{k-j-1}\binom{k-j-1}{i}A^{\alpha_1,\cdots,\alpha_k}_{,\alpha_{i+j+1},\cdots,\alpha_{k-1}}n_{\alpha_k}.$$

DEFINITION. Assume that $F\in C^{l+1}(R^\eta)$; $u\in U^{(2l)}$; $\partial G\in C^m$; $A^{\alpha_1,\cdots,\alpha_k}\in C^k(\overline{G})$, for $0\leqq k\leqq m$. If there exists a function $v\in C^m(\overline{G})$, that satisfies: (1°) $L^*(v(x))=[F]_{,u}$, $x\in G$, (2°) $\delta J(u;\delta u)$, as given in (3.1), vanishes for all functions $\delta u\in C^m(\text{nbh }\partial G)$, then this function v is called *a variational adjoint of u with respect to L and J*. The boundary conditions (2°), or conditions that are equivalent are called the *variational adjoint boundary conditions.*

THEOREM 2 (MAIN THEOREM). *If v is a variational adjoint of u, with respect to L and J, then the variational adjoint boundary conditions on* $\partial G'$, *the noncharacteristic part of* ∂G, *are given by* (i), (ii) *and* (iii) *as follows:*

(i) $\quad(\partial/\partial n)^jv(x)=0,$ *for* $0\leqq j\leqq m-l-1$ *and* $x\in\partial G'$.

(If $l=m$, *then this condition is vacuous.)*

(ii) $(\partial/\partial n)^{m-l}v(x)=(-1)^{m-l}(n_{\alpha_1}\cdots n_{\alpha_m}\overset{\alpha_1,\cdots,\alpha_m}{A(x)})^{-1}n_{\alpha_1}\cdots n_{\alpha_{l-1}}Q^{\alpha_1,\cdots,\alpha_{l-1}}$, *for* $x\in\partial G'$.

(iii)
$$\sum_{h=0}^{l-1-j} \lambda(h;j)_{\alpha_1,-,\alpha_{j+h}} \sum_{i=m-l}^{m-j-h-1} v_{,\alpha_{j+h+1},-,\alpha_{j+h+i}} P(i;j+h)^{\alpha_1,-,\alpha_i+j+h}$$
$$= \sum_{h=0}^{l-1-j} \lambda(h;j)_{\alpha_1,-,\alpha_{j+h}} Q^{\alpha_1,-,\alpha_{j+h}},$$

for $x \in \partial G'$ *and* $l-2 \geqq j \geqq 0$. *(If* $l=1$, *then this condition is vacuous.)*
For $0 \leqq j \leqq l-2$, $0 \leqq h \leqq l-1-j$, $x \in \partial G$, *the*

$$\lambda(h;\, j)s^{\alpha_1,-,\alpha_{j+h}}$$

denote linear differential expressions of order h.

DEFINITION. Each of the conditions in Theorem 2 is called *variational adjoint boundary condition* k, if the highest order of differentiation of v in that condition is k. So $k=j$ in (i), $k=m-l$ in (ii) and $k=m-j-1$ in (iii).

4. Necessary conditions.

THEOREM 3 (ELLIPTIC CASE). *Let the following conditions be satisfied for some integer* $q \geqq \max\{0,\ 2l+[\nu/2]+1-m\}$, $2l \in [2,\ m]$, *and* $t=p+[\nu/2]+1$. (1°) G *is bounded and* $\partial G \in C^{m+t}$. (2°) L *is uniformly strongly elliptic in* $\overline{G}$. (3°) $A^{\alpha_1,-,\alpha_j} \in C^{2j+t}(\overline{G})$, $0 \leqq j \leqq m$. (4°) $f \in C^t(\overline{G})$. (5°) $F \in C^{l+[\nu/2]+2}(R^\eta)$. (6°) $W = U^{(m+q)}$. *If now* $u \in W$, *then the Fredholm alternative holds for the boundary value problem*

$$(4.1)\qquad \begin{aligned} L^*(v(x)) &= [F]_{,u}, & x &\in G, \\ (\partial/\partial n)^j v(x) &= 0, & 0 &\leqq j \leqq \tfrac{1}{2}m-1,\ x \in \partial G, \end{aligned}$$

while any solution of (4.1) *is of class* $C^m(\overline{G})$. *If moreover* $\delta J(u;\, \delta u)=0$, *for all* $\delta u \in W_H$, *then any solution of* (4.1) *also satisfies the variational adjoint boundary conditions* $\frac{1}{2}m, -, m-1$. ($\partial G = \partial G'$.) *Conversely, if there exists a solution of* (4.1) *that satisfies the variational adjoint boundary conditions* $\frac{1}{2}m, -, m-1$, *then* $\delta J(u;\, \delta u)=0$, *for all* $\delta u \in W_H$.

DEFINITION. If the requirements (1°)–(6°) of Theorem 3 are met then the associated boundary value problem is defined by

(4.2)
$$L(u(x)) = f(x),\ u \in W,\ x \in G,$$
$$L^*(v(x)) = [F]_{,u},\ x \in G.$$
The variational adjoint boundary conditions 0,—,$m-1$, as given in Theorem 2, are satisfied on ∂G.

COROLLARY. *If* J *has a relative extremum within* W *for some* $u \in W$, *if the requirements* (1°)–(6°) *of Theorem* 3 *are met, and if* $[F]_{,u}$ *is orthogonal in* $L_2(G)$ *to the null space of* L, *then there exists at least one function* $v \in C^m(\bar{G})$, *such that* u *and* v *together satisfy the associated boundary value problem.*

With regard to the *domain* G, the following definitions are in force in the next theorem. $G_{\nu-1}$ is an open domain in $R^{\nu-1}$ with boundary $\partial G_{\nu-1}$. $H_{\nu-1}$ is the intersection of the $R^{\nu-1}$-closure of $G_{\nu-1}$ with an open $R^{\nu-1}$-neighborhood of $\partial G_{\nu-1}$. $G = G_{\nu-1} \times (0, T)$. ∂G is the boundary of G. $\partial G(0) = \{x \in R^\nu : x' \in G_{\nu-1} \wedge x_\nu = 0\}$. ($x' = (x_1, —, x_{\nu-1})$.) Similarly for $\partial G(T)$. $\partial H(0) = \{x \in R^\nu : x' \in H_{\nu-1} \wedge x_\nu = 0\}$. Similarly for $\partial H(T)$. $\partial G(0, T] = \{x \in R^\nu : x' \in \partial G_{\nu-1} \wedge x_\nu \in (0, T]\}$.

THEOREM 4 (PARABOLIC CASE). *Let the following conditions be satisfied for*: $r = 1 + [(2q + \nu + 1)/2m]$; $s = m + q + r + [\nu/2]$; $t = s - r + m(r+1)$; $t(0) = 3m + [\nu/2] + m[(\nu+1)/2m]$. (1°) $l \subset [1, m/2]$, $q \geqq t(0) + 2l - m$. (2°) $G_{\nu-1}$ *and* T *are bounded*; $\partial G_{\nu-1} \in C^{m(r+1)}$. (3°) $\overset{\alpha_1, —, \alpha_k}{A(x)} \equiv 0$, *if at least one index has value* ν *and* $k \in [2, m]$, *and* $A^\nu \equiv 1$. *Furthermore,* L *is uniformly parabolic in* $\bar{G}$. (4°) $\overset{\alpha_1, —, \alpha_j}{A(x)} \in C^{j+s}(\bar{G})$, $0 \leqq j \leqq m$. (5°) $f \in C^t(\bar{G})$; $(\partial/\partial \mathbf{n})^j f(x) = 0$, *for* $0 \leqq j \leqq t$ *and* $x \in \partial H(0)$. (6°) $F \in C^{l+1+t(0)}(R^\eta)$. (7°) $W = U^{(m+q)}$. (8°) $(\partial/\partial \mathbf{n})^j [F]_{,u} = 0$, *for* $0 \leqq j \leqq t(0)$, $x \in \partial H(0)$, *and* $u \in W$. (9°) $\partial F/\partial u_{,\nu}$ *has compact support in* $\partial G(T)$ *for any* $u \in W$. (10°) $\partial F/\partial u_{,\alpha_1, —, \alpha_j} \equiv 0$, *for* $2 \leqq j \leqq l$ *and* $x \in \text{nbh } \partial G(0) \cup \partial G(T)$. *If now* $u \in W$, *then the initial boundary value problem*

$$
\begin{aligned}
L^* v(x) &= [F]_{,u}, \quad \text{for } x \in G, \\
(\partial/\partial \mathbf{n})^j v(x) &= 0, \quad \text{for } 0 \leqq j \leqq \tfrac{1}{2}m - 1 \text{ and } x \in \partial G[0, T), \qquad (4.3) \\
v(x) &= + \partial F/\partial u_{,\nu}, \quad \text{for } x \in \partial G(T),
\end{aligned}
$$

admits a unique solution, which is of class $C^m(\bar{G})$. *If moreover* $\delta J(u; \delta u) = 0$, *for all* $\delta u \in W_H$, *then a solution of* (4.3) *also satisfies the variational adjoint boundary conditions* $\frac{1}{2}m, —, m-1$ *on the lateral boundary,* $\partial G(0, T)$ *and the condition*: $v(x) = +\partial F/\partial u_{,\nu}$ *on* $\partial G(0)$. *Conversely, if there exists a solution of* (4.3) *that also satisfies these extra boundary conditions then* $\delta J(u; \delta u)$ *vanishes for all* $\delta u \in W_H$.

REMARK. The associated boundary value problem is defined similarly to that in Theorem 3. A corollary similar to that with Theorem 3 is valid for this problem.

The existence and regularity statements in Theorems 3 and 4 are based on results in [2]. Simple results for the hyperbolic case are omitted in this announcement. They can be found in [1].

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