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EXTENSION OF POSITIVE HOLOMORPHIC LINE BUNDLES

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In this note, we announce a result on extending complex line bundles through subvarieties of codimension 2. The motivation for this result is that it allows us to extend a recent result of Phillip Griffiths [2] on meromorphically extending holomorphic maps into compact Kahler manifolds. Details and further related results will appear elsewhere.

A holomorphic line bundle L on a complex manifold M is said to be *semipositive* if there exists a hermitian metric h on L such that the curvature form

$$\Omega = \frac{i}{2\pi} \partial\bar{\partial} \log h$$

is positive semidefinite at all points of M (i.e., the locally defined functions $\log h$ are plurisubharmonic).

THEOREM. *Let M be a complex manifold, and let S be an analytic set in M such that* codim $S=2$. *Then every semipositive holomorphic line bundle L on $M-S$ extends to a holomorphic line bundle on M.*

If codim $S \geqq 3$, then it is a well-known fact that any line bundle L on $M-S$ extends to M (see [3]).

In order to prove the theorem, one must show that L induces the zero element of $H_S^2(M, \mathcal{O}^*) \approx \Gamma(M, \mathcal{H}_S^2\mathcal{O}^*)$. Therefore it suffices to show that L extends locally, and the theorem is then a consequence of the following lemma applied to the curvature form Ω.

We let D denote the open unit disk in $\mathbf{C}$.

LEMMA. *Let*

$$\omega = i \sum f_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta \qquad (1 \leqq \alpha, \beta \leqq n)$$

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be a real closed (1, 1)*-form on the domain*

$$W = (D^2 - 0) \times D^{n-2} \subset \mathbf{C}^n.$$

If $f_{11} \geqq 0$ *and* $f_{22} \geqq 0$ *on* W, *then there exists a real-valued function* u *on* W *such that* $\omega = dd^c u$. *In particular, if* ω *is a Kahler form on* W, *then* $\omega = dd^c u$, *where* u *is a function on* W.

Write $W = W_1 \cup W_2$, where

$$W_j = \{z \in W : z_j \neq 0\}, \quad \text{for } j = 1, 2.$$

Then $\omega | W_j = dd^c u_j$, where u_j is a real-valued function on W_j ($j = 1, 2$). Let $h = u_1 - u_2$ on $W_1 \cap W_2$. Then $dd^c h = 0$, i.e., h is pluriharmonic. For the case $n = 2$, u_1 and u_2 are subharmonic in each variable separately. The main point of the lemma (and the theorem) is that we can then write $h = h_1 - h_2$, where h_j is a pluriharmonic function on W_j ($j = 1, 2$), and therefore $u = u_j - h_j$ is a globally defined function on W with $\omega = dd^c u$. The proof uses the solution of the Dirichlet problem on the annulus $A_r = \{r < |z| < 1\}$. By considering the biannulus $A_r \times A_s$ and letting $(r, s) \to (0, 0)$, one constructs functions $\tilde{u}_j$ on W_j ($j = 1, 2$) such that $\tilde{u}_1$ and $\tilde{u}_2$ are harmonic in each variable separately, and $h = \tilde{u}_1 - \tilde{u}_2$. The existence of h_1 and h_2 then follows from the equation $h = \tilde{u}_1 - \tilde{u}_2$.

In [2], Phillip Griffiths proved the following result.

THEOREM (GRIFFITHS). *Let* $f: D^n - 0 \to X$ *be a holomorphic map, where* X *is a compact Kahler manifold. If* $n \geqq 3$, *then* f *extends meromorphically to* D^n (*i.e., the closure of the graph of* f *is an analytic set in* $D^n \times X$).

Griffiths' idea is to apply a theorem of Errett Bishop [1], [4] on extending analytic sets with finite volume. Consider the Kahler form

$$\omega = \frac{i}{2} \sum dz_\alpha \wedge d\bar{z}_\alpha + f^* \omega_X$$

on $D^n - 0$, where ω_X is the given Kahler form on X. The volume of the graph of f is then given by $\int \omega^n$. Since $H^2(D^n - 0, \mathbf{R}) = 0$ and $H^1(D^n - 0, \mathcal{O}) = 0$, for $n \geqq 3$, one can write $\omega = dd^c u$, where u is a plurisubharmonic function on $D^n - 0$. By approximating u by smooth plurisubharmonic functions on a ball $B \subset\subset D^n$ about 0 and by applying Stokes' theorem, Griffiths concludes that $\int_B (dd^c u)^n < +\infty$.

By the above lemma, we can also write $\omega = dd^c u$ for the case $n = 2$ (although $H^1(D^2 - 0, \mathcal{O}) \neq 0$). Therefore, Griffiths' theorem is also valid for $n = 2$.

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