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# HOMOTOPY TYPES OF SOME PL COMPLEXES[1]

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Let $\mathrm{PL}_n$ represent the semisimplicial group of germs of piecewise linear (PL), origin-preserving homeomorphisms of $R^n$. Since this is the structure group for $n$-dimensional PL-bundles ([9], [8]), knowledge of its homotopy type is important for classifying PL structures. Beautiful work has been done in the stable range, making the intrinsic relations between differentiable, piecewise linear and topological categories very clear, but very little is known in the nonstable case. Kuiper and Lashof [8] have treated the nonstable results from a unified point of view.

The purpose of this note is to announce the computation of the homotopy groups of certain PL embedding spaces and the determination of the homotopy types of $\mathrm{PL}_2$ and $\mathrm{PL}_3$ based on the techniques developed in [8]. Some applications are also indicated.

The author wishes to express his gratitude to many people, among whom he is especially pleased to mention J. Milnor, M. Kato, C. Rourke and I. Tamura, for helpful, stimulating, and encouraging discussions and conversations.

We will work in the piecewise linear category. We denote as usual by $\Delta_k$, $S^m$, $I^p$, $\partial I^p$ and $O_n$ the standard $k$-simplex, $m$-sphere, $p$-cube, the boundary of the $p$-cube and the semisimplicial (s.s.) $n$-dimensional orthogonal group respectively. Let $\mathcal{E}(I^p, I^p\times S^n)$ denote an s.s. complex whose typical $k$-simplex is a $k$-isotopy $f:\Delta_k\times I^p\to\Delta_k\times I^p\times S^n$ satisfying:

(i) $f|\Delta_k\times\partial I^p=\text{identity}$,

(ii) $f$ is extendable to $\Delta_k\times I^p\times S^n$ as a $k$-isotopy.

Here $\Delta_k\times I^p$ is identified with $\Delta_k\times I^p\times q$, where $q$ is the base point of $S^n$.

THEOREM.[2] *$\pi_i(\mathcal{E}(I^p,I^p\times S^n))\cong\pi_{i+p}(S^n)$ for*

(a) *$p=1$ and all $n$,*

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[1] The work announced here is contained in the author's doctoral thesis submitted to M.I.T.

[2] This contradicts the announcement of K. C. Millett.

(b) $p=2$ *and* $n=1$.

REMARK. We can easily construct a homomorphism from $\pi_i(\mathcal{E}(I^p, I^p\times S^n))$ onto $\pi_{i+p}(S^n)$. To show that this homomorphism is injective, we need the unknotting of balls in codimension three [**15**] for $n\geqq 3$, and the Schoenflies theorem [**10**] in the rest of the cases. In both of these cases we use the $k$-isotopy extension theorem of Hudson [**6**] and the celebrated Alexander Trick.

COROLLARY.[3] $\mathrm{PL}_2$ *is homotopy equivalent to* $O_2$.

This is clear from the isomorphism $\pi_i(\mathrm{PL}_2)\cong\pi_{i-1}(\mathcal{E}(I, I\times S^1))$, [**8**], $(i\geqq 1)$, and (a) of the above theorem.

REMARK. This is also obtained by G. P. Scott [**12**] and C. Morlet [**11**] independently.

COROLLARY 2. $\mathrm{PL}_3$ *is homotopy equivalent to* $O_3$.

For $i\geqq 2$ we have $\pi_{i+1}(\mathrm{PL}_3)\cong\pi_i(\mathcal{E}(I, I\times S^2))\cong\pi_{i+1}(S^2)$ using (a) and (b) of the above theorem. We compute $\pi_i(\mathrm{PL}_3)$ for $i=0$, 1 and 2 separately using the stable results.

REMARK. G. P. Scott [**13**] has also obtained this result independently.

COROLLARY 3. *A nonzero cross-section of an $R^n$-bundle over a finite complex determines an $R^1$-subbundle containing the section, uniquely (up to an isomorphism of the original bundle, fixing the subbundle).*

The result of Browder [**2**], together with this corollary, implies that an $R^1$-subbundle of a PL $R^n$-bundle is not generally a direct summand.

Let $\mathrm{Aut}^{\mathrm{Diff}}(S^n)$ denote a semisimplicial complex whose typical $k$-simplex is an orientation preserving $C^\infty$-diffeomorphism of $\Delta_k\times S^n$ commuting with the projection onto the first factor. Using the results of Morlet [**11**] or C. Rourke (unpublished) we have:

COROLLARY 4. $\mathrm{Aut}^{\mathrm{Diff}}(S^n)$ *is homotopy equivalent to* $SO_{n+1}$ $(n\leqq 3)$.

REMARK. In particular this implies Cerf's result [**4**] that $\Gamma_4=0$.

The case $n=2$ was proved by Smale [**14**] in a stronger form, while $n=3$ gives an affirmative answer to the so-called Smale conjecture.

Let $\mathrm{Top}_n$ denote the semisimplicial complex of (topological) homeomorphisms of $R^n$ keeping the origin fixed. Using Morlet's results again (or Cerf [**3**]), we obtain:

COROLLARY 5. $\mathrm{Top}_n$ *is homotopy equivalent to* $\mathrm{PL}_n$ *for* $n\leqq 3$.

[3] Lemma 1(ii) of [1] is false. The result here is proved by different methods.

REMARK. The case $n=2$ together with Corollary 1 implies a weaker version of Kneser's theorem [7].

Finally, following Hirsch [5] we have:

COROLLARY 6. *There exists a* PL *submanifold* $M^4$ *in* $S^7$ *without (topological) normal* $R^n$*-bundle.*

## BIBLIOGRAPHY

1. T. Akiba, *On the homotopy type of* $\mathrm{PL}_2$, J. Fac. Sci. Univ. Tokyo **14** (1967), 197–204.

2. W. Browder, *Open and closed disk bundles*, Ann. of Math. (2) **83** (1966), 218–230. MR **32** #6485.

3. J. Cerf, *Groupes d'homotopie locaux et groupes d'homotopie mixtes des espaces bitopologiques. Presque n-locale connexion. Applications*, C. R. Acad. Sci. Paris **253** (1961), 363–365. MR **25** #3530.

4. ———, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4=0$), Lecture Notes in Math., no. 53, Springer-Verlag, Berlin and New York, 1968. MR **37** #4824.

5. M. W. Hirsch, *On tubular neighborhoods of piecewise linear and topological manifolds*, Conference on Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 63–80. MR **38** #5226.

6. J. F. P. Hudson, *Extending piecewise-linear isotopies*, Proc. London Math. Soc. (3) **16** (1966), 651–668. MR **34** #2020.

7. H. Kneser, *Die Deformationssätze der einfach zusammenhängenden Flachen*, Math. Z. **25** (1926), 362–372.

8. N. H. Kuiper and R. K. Lashof, *Microbundles and bundles*. II. *Semisimplicial theory*, Invent. Math. **1** (1966), 243–259. MR **35** #7340.

9. J. Milnor, *Microbundles and differentiable structures*, Princeton University, Princeton, N. J., 1961 (mimeographed).

10. E. E. Moise, *Affine structures in 3-manifolds*. II. *Positional properties of 2-spheres*, Ann. of Math. (2) **55** (1952), 172–176. MR **13**, 574.

11. C. Morlet, *Lissage des homéomorphismes*, C.R. Acad. Sci. Paris. Sér. A-B **268** (1969), A1323-A1326. MR **39** #6333.

12. G. P. Scott, *Thesis*, University of Warwick, 1969.

13. ———, *On the homotopy type of* $\mathrm{PL}_3$, University of Liverpool, 1970 (mimeographed).

14. S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626. MR **22** #3004.

15. E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Études Sci., Paris, 1963 (mimeographed notes).

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