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THE INTERPOLATORY BACKGROUND OF THE EULER-MACLAURIN QUADRATURE FORMULA

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In [4] the first named author discussed the explicit solutions of the cubic spline interpolation problems. We are now concerned with quintic spline functions. Let $S_5[0, n]$ denote the class of quintic spline functions $S(x)$ defined in the interval $[0, n]$ and having the points $0, 1, \cdots, n-1$ as knots. This means that the restriction of $S(x)$ to the interval $(\nu, \nu+1)$ $(\nu=0, \cdots, n-1)$ is a fifth degree polynomial, and that $S(x) \in C^4[0, n]$. With these functions we can solve uniquely the following three types of interpolation problems.

1. *Natural quintic spline interpolation.* We are required to find $S(x) \in S_5[0, n]$ such as to satisfy the conditions

$$S(\nu) = f(\nu) \qquad (\nu = 0, \cdots, n), \tag{1}$$

$$S'''(0) = S^{(4)}(0) = S'''(n) = S^{(4)}(n) = 0. \tag{2}$$

2. *Complete quintic spline interpolation.* We are to find $S(x) \in S_5[0, n]$ so as to satisfy the conditions

$$S(\nu) = f(\nu) \qquad (\nu = 0, \cdots, n), \tag{3}$$

$$S'(0) = f'(0), \quad S''(0) = f''(0), \quad S'(n) = f'(n), \quad S''(n) = f''(n). \tag{4}$$

3. *The interpolation of Euler-Maclaurin data.* Here we seek $S(x) \in S_5[0, n]$ such that

$$S(\nu) = f(\nu) \qquad (\nu = 0, \cdots, n), \tag{5}$$

$$S'(0) = f'(0), \quad S'''(0) = f'''(0), \quad S'(n) = f'(n), \quad S'''(n) = f'''(n). \tag{6}$$

In the present note we propose to do for quintic spline interpolation what was done in [4] for cubic interpolation. Also the method used is similar; in the present case we derive our results from the 5th degree case of cardinal spline interpolation discussed in [2]. We describe here the results concerning the third problem (5) and (6).

The foundation of our discussion is the quintic B-spline

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$$
\begin{aligned}
M(x) = M_6(x) = \frac{1}{5!}\{&(x+3)_+^5 - 6(x+2)_+^5 + 15(x+1)_+^5 \\
&- 20x_+^5 + 15(x-1)_+^5 - 6(x-2)_+^5 + (x-3)_+^5\},
\end{aligned} \tag{7}
$$

where $x_+ = \max(0, x)$. Evidently $M(x)$ is a quintic spline function with knots at $-3, -2, -1, 0, 1, 2, 3$, and having its support in $[-3, 3]$.

LEMMA 1. *Every $S(x) \in S_5[0, n]$ admits a unique representation of the form*

$$
S(x) = \sum_{-2}^{n+2} C_j M(x-j) \qquad (0 \leqq x \leqq n). \tag{8}
$$

The existence and unicity of the solution of problem 3 (see [1]) implies that we may write the solution in the form

$$
\begin{aligned}
S(x) = \sum_{0}^{n} f(\nu) L_\nu(x) + f'(0)\Lambda_1(x) + f'''(0)\Lambda_3(x) \\
- f'(n)\Lambda_1(n-x) - f'''(n)\Lambda_3(n-x),
\end{aligned} \tag{9}
$$

where the coefficients of the data are the corresponding fundamental functions that are uniquely defined by appropriate unit-data. By Lemma 1 we may represent these fundamental functions as follows:

$$
L_\nu(x) = \sum_{-2}^{n+2} c_{j,\nu} M(x-j) \qquad (\nu = 0, \cdots, n), \tag{10}
$$

$$
\Lambda_1(x) = \sum_{-2}^{n+2} c_{j,-1} M(x-j), \qquad -\Lambda_1(n-x) = \sum_{-2}^{n+2} c_{j,n+1} M(x-j), \tag{11}
$$

$$
\Lambda_3(x) = \sum_{-2}^{n+2} c_{j,-2} M(x-j), \qquad -\Lambda_3(n-x) = \sum_{-2}^{n+2} c_{j,n+2} M(x-j), \tag{12}
$$

with coefficients that are yet to be determined.

Introducing the representation (8) into the equations (5) and (6), we obtain a system of $n+5$ equations for the $n+5$ unknown coefficients C_j. We denote the inverse of the matrix of this linear system by

$$
\Gamma_2 = \|c_{j,\nu}\| \qquad (j, \nu = -2, -1, \cdots, n+2). \tag{13}
$$

The determination of this matrix depends on the four algebraic integers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which are the zeros of the fifth Euler-Frobenius polynomial

$$
\Pi_5(x) = x^4 + 26x^3 + 66x^2 + 26x + 1 \tag{14}
$$

(for the generating function of the polynomials $\Pi_n(x)$ and their name see [3]). These zeros are reciprocal in pairs and satisfy the inequalities

$$\lambda_4 < \lambda_3 < -1 < \lambda_2 < \lambda_1 < 0.$$

Setting $x+x^{-1}=z$, the reciprocal equation $\Pi_5(x)=0$ reduces to the quadratic $z^2+26z+64=0$ having the roots $-13\pm(105)^{1/2}$. It follows that

$$\text{(15)} \qquad \lambda_1+\lambda_1^{-1} = -13-(105)^{1/2}, \qquad \lambda_2+\lambda_2^{-1} = -13+(105)^{1/2}.$$

The solution of our problem depends in the first place on the so-called fundamental cardinal spline function $L(x)$ satisfying the relations

$$L(0) = 1, \qquad L(j) = 0 \quad \text{if } j \neq 0.$$

We find that

$$L(x) = \alpha \sum_{-\infty}^{\infty} \lambda_1^{|j|} M_6(x-j) + \beta \sum_{-\infty}^{\infty} \lambda_2^{|j|} M_6(x-j),$$

where

$$\alpha^{-1} = -(\lambda_1-\lambda_1^{-1})(105)^{1/2}/60, \qquad \beta^{-1} = -(\lambda_2-\lambda_2^{-1})(105)^{1/2}/60.$$

The fundamental functions (10), (11), and (12), may now be expressed as appropriate linear combinations of $L(x)$ and of the four eigensplines

$$S_\nu(x) = \sum_{-\infty}^{\infty} \lambda_\nu^j M_6(x-j) \qquad (\nu = 1,2,3,4).$$

It is clear a priori that the elements of the matrix (13) are rational numbers. Actually, the elements of (13) can be explicitly expressed in terms of certain sequences of integers defined by appropriate recurrence relations. We define two even sequences (a_k) and (b_k) of integers such that

$$\text{(16)} \qquad \lambda_1^k+\lambda_1^{-k} = a_k - b_k(105)^{1/2}, \qquad \lambda_2^k+\lambda_2^{-k} = a_k + b_k(105)^{1/2}.$$

These sequences may also be defined as solutions of the recurrence relation

$$\text{(17)} \qquad x_{k+4}+26x_{k+3}++66x_{k+2}+26x_{k+1}+x_k=0 \qquad (-\infty<k<\infty),$$

with the initial values

$$\text{(18)} \qquad a_{-2} = 272, \quad a_{-1} = -13, \quad a_0 = 2, \quad a_1 = -13,$$

and

$$b_{-2} = -26, \quad b_{-1} = 1, \quad b_0 = 0, \quad b_1 = 1, \tag{19}$$

respectively. We define two further sequences by

$$A_k = a_{k+1} - a_{k-1}, \qquad B_k = b_{k+1} - b_{k-1}. \tag{20}$$

We may now state

THEOREM 1. *In terms of the sequences* (a_k), (b_k), (A_k), (B_k), *defined by the relations* (17) *to* (20), *we may write*

$$c_{j,0} = \frac{120}{A_n^2 - 105B_n^2}\,(B_n a_{n-|j|} - A_n b_{n-|j|}), \tag{21}$$

while

$$c_{j,n} = c_{n-j,0}. \tag{22}$$

Furthermore,

$$c_{j,\nu} = \frac{120}{A_n^2 - 105B_n^2}\left\{B_n(a_{n-|\nu-j|} + a_{n-\nu-j}) - A_n(b_{n-|\nu-j|} + b_{n-\nu-j})\right\}, \tag{23}$$

if $0 < \nu < n$,

$$c_{j,-1} = \frac{2}{A_n^2 - 105B_n^2}\left\{(15B_n - A_n)a_{n-j} + 15(7B_n - A_n)b_{n-j}\right\}, \tag{24}$$

$$c_{j,-2} = \frac{1}{6(A_n^2 - 105B_n^2)}\left\{(A_n - 3B_n)a_{n-j} + 3(A_n - 35B_n)b_{n-j}\right\}. \tag{25}$$

The remaining coefficients are given by

$$c_{j,n+1} = -c_{n-j,-1}, \tag{26}$$

$$c_{j,n+2} = -c_{n-j,-2}. \tag{27}$$

Besides (22), (26), *and* (27), *we have the symmetry relations*

$$c_{j,\nu} = c_{n-j,n-\nu}, \quad \textit{for all } j,\ 0 \leqq \nu \leqq n. \tag{28}$$

We may also express our results as follows: The spline function $S(x)$ satisfying the relations (5) and (6) is given by (8), where

$$\begin{aligned} C_j = {} & c_{j,-2}f'''(0) + c_{j,-1}f'(0) + \sum_{\nu=0}^{n} c_{j,\nu}f(\nu) \\ & + c_{j,n+1}f'(n) + c_{j,n+2}f'''(n) \quad (j = -2, -1, \cdots, n+2). \end{aligned} \tag{29}$$

Here the coefficients $c_{j,\nu}$ are the elements of the matrix (13) and their values are described by Theorem 1.

As a numerical example we choose $n = 2$ and find

$$\Gamma_2 = \frac{1}{16} \cdot \begin{Vmatrix} -1621/60 & -56 & 19 & -52 & 49 & -22 & 61/60 \\ 189/60 & -43 & -26 & 68 & -26 & 11 & -29/60 \\ -61/60 & 22 & 49 & -52 & 19 & -8 & 21/60 \\ 29/60 & -11 & -26 & 68 & -26 & 11 & -29/60 \\ -21/60 & 8 & 19 & -52 & 49 & -22 & 61/60 \\ 29/60 & -11 & -26 & 68 & -26 & 43 & -189/60 \\ -61/60 & 22 & 49 & -52 & 19 & 56 & 1621/60 \end{Vmatrix}.$$

The problem 3 hereby solved was referred to as concerning the *Euler-Maclaurin data* for the following reason. From the results of [1] it is clear that if we integrate the interpolating spline function (9) between the limits 0 and n we obtain the relation

$$\begin{aligned} \int_0^n f(x)\,dx = \frac{1}{2} f(0) + f(1) + \cdots + \frac{1}{2} f(n) + \frac{1}{12}(f'(0) - f'(n)) \\ - \frac{1}{720}(f'''(0) - f'''(n)), \end{aligned} \tag{30}$$

which is the Euler-Maclaurin quadrature formula for our data. The reason for this is that our interpolation process, as well as the quadrature formula (30), are both exact for the class of spline functions $S_5[0, n]$, and that both are uniquely characterized by this property. This connection also explains the title of the present note.

Among our three interpolation problems the third is the most readily accessible by our method. From its solution similar explicit results can be derived for the first two problems. Our approach also generalizes to heptic and higher odd-degree spline interpolation problems.

In conclusion let us point out that if we let $n \to \infty$, then the matrix (13) converges rapidly to an infinite matrix Γ^+ whose elements are the coefficients corresponding to the semicardinal quintic spline interpolation of the data $f'''(0), f'(0), f(0), f(1), f(2), \cdots$ (see [4] for the corresponding cubic case).

References

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