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WEIGHTED APPROXIMATION OF CONTINUOUS FUNCTIONS[1]

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1. **Notation.** Let X be a completely regular Hausdorff space and E a (real or complex) locally convex Hausdorff space. $F(X, E)$ is the vector space of all mappings from X into E, and $C(X, E)$ is the vector subspace of all such mappings that are continuous. $B_\infty(X, E)$ is the vector subspace of $F(X, E)$ consisting of those bounded f that vanish at infinity. The vector subspace $C(X, E)\cap B_\infty(X, E)$ is denoted by $C_\infty(X, E)$. If X is locally compact, $\mathcal{K}(X, E)$ will denote the subspace of $C(X, E)$ consisting of those functions that have compact support. The corresponding spaces for $E=\boldsymbol{R}$ or $\boldsymbol{C}$ are written omitting E. A *weight* v on X is a nonnegative upper semicontinuous function on X. A directed family of weights on X is a set of weights on X such that given $u, v\in V$ and $\lambda\geqq 0$ there is a $w\in W$ such that $\lambda u, \lambda v\leqq w$. If U and V are two directed families of weights on X and for every $u\in U$ there is a $v\in V$ such that $u\leqq v$, we write $U\leqq V$. If V is a directed family of weights on X, the vector space of all $f\in F(X, E)$ such that $vf\in B_\infty(X, E)$, for any $v\in V$, is denoted by $FV_\infty(X, E)$ and is called a weighted function space. On $FV_\infty(X, E)$ we shall consider the topology determined by all the seminorms $f\mapsto \sup\{v(x)p(f(x)); x\in X\}$ where $v\in V$ and p is a continuous seminorm on E. $CV_\infty(X, E)$ will denote the subspace $FV_\infty(X, E)\cap C(X, E)$, equipped with the induced topology. The weighted function spaces $CV_\infty(X, E)$ will be called *Nachbin spaces.*

2. **Completeness properties of Nachbin spaces** [6]. If for every $x\in X$ there is a weight $u\in U$ such that $u(x)>0$, we write $U>0$.

LEMMA. *If E is complete and $U>0$, then $FU_\infty(X, E)$ is complete.*

THEOREM 1. *Suppose that E is complete, and U and V are two directed families of weights on X with $U\leqq V$. If $V>0$ on X and $CU_\infty(X, E)$ is closed in $FU_\infty(X, E)$, the Nachbin space $CV_\infty(X, E)$ is complete.*

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In case E is $\boldsymbol{R}$ or $\boldsymbol{C}$, the above theorem was obtained by Summers, under the hypothesis that $U>0$ on X. (See Theorem 3.6 of [**10**].)

THEOREM 2. *Suppose that E is complete and U and V are two directed families of weights on X with $U\leqq V$. If $V>0$ on X and $CU_\infty(X, E)$ is quasi-complete, the Nachbin space $CV_\infty(X, E)$ is quasi-complete.*

3. **Dual spaces** [**6**]. Throughout this paragraph X will be a locally compact Hausdorff space. In this case, for any set of weights V on X, the space $\mathcal{K}(X, E)$ is densely contained in $CV_\infty(X, E)$. In fact, even $\mathcal{K}(X)\otimes E$ is densely contained in $CV_\infty(X, E)$. Let E'_w denote the topological dual of E endowed with the topology $\sigma(E', E)$. An E'_w-valued bounded Radon measure u on X is a continuous linear mapping u from $\mathcal{K}(X)$ into E'_w when $\mathcal{K}(X)$ is endowed with the topology of uniform convergence on X. Following Grothendieck [**4**], an E'_w-valued bounded Radon measure u on X is called *integral* if the linear form L defined over $\mathcal{K}(X)\otimes E$ by $L(\sum \phi_i\otimes y_i)=\sum \langle y_i, u(\phi_i)\rangle$ is continuous in the topology induced by $C_\infty(X, E)$, in which case it can be uniquely continuously extended to $C_\infty(X, E)$. Let $L\in C_\infty(X, E)'$; if we define $u(\phi)$ for each $\phi\in\mathcal{K}(X)$ by $\langle y, u(\phi)\rangle = L(\phi\otimes y)$ for all $y\in E$, then u is an E'_w-valued bounded Radon measure. The transpose u' of u is a linear map from E into $M_b(X)$, the space of all bounded Radon measures on X. For every $y\in E$ there corresponds a unique regular Borel measure μ_y such that $\mu_y(B)=\langle u'(y), \chi_B\rangle$, for all Borel subsets B of X. There exists a continuous seminorm p on E and a constant $k>0$ such that $|L(f)|\leqq k\|f\|_p$ for all $f\in C_\infty(X, E)$. Hence $|\langle y, u(\phi)\rangle| = |L(\phi\otimes y)| \leqq kp(y)\|\phi\|_\infty$. Thus, the bounded Radon measure $u'(y)$ has norm $\|u'(y)\|\leqq kp(y)$, and the corresponding Borel measure μ_y is such that $|\mu_y(B)|\leqq\|\mu_y\|\leqq kp(y)$. This shows that, for a fixed Borel subset $B\subset X$, the map $y\mapsto\mu_y(B)$ belongs to E'. Call this map $\mu(B)$. The set function $B\mapsto\mu(B)$, defined on the σ-ring of all Borel subsets of X and with values on E', is countably additive. For any finite families $\{B_i\}_{i\in I}$ of disjoint Borel subsets of X, whose union is X, and $\{y_i\}_{i\in I}$ of elements of E with $p(y_i)\leqq 1$ for each $i\in I$, we have

$$(*)\qquad \left|\sum_{i\in I}\langle y_i, \mu(B_i)\rangle\right| \leqq k.$$

An E'_w-valued bounded Radon measure u on X such that the corresponding set function μ satisfies (*) for some continuous seminorm p on E and some constant $k>0$ is said to have *finite p-semivariation.* On the other hand, following Dieudonné [**2**], an E'_w-valued bounded

Radon measure on X is said to be *p-dominated* if there is a positive bounded Radon measure μ on X such that $|\langle y, u(\phi)\rangle| \leqq \mu(|\phi|)p(y)$ for all $y \in E$ and $\phi \in \mathcal{K}(X)$. The arguments contained in Singer [9] and Các [1] can be extended to prove the following:

LEMMA. *Let u be an E'_w-valued bounded Radon measure on X. The following are equivalent:*

(a) *u is integral;*

(b) *u is p-dominated, for some continuous seminorm p on E;*

(c) *u has finite p-semivariation, for some continuous seminorm p on E.*

We denote by $M_b(X, E')$ the vector space of all E'_w-valued bounded Radon measures on X which satisfy (a) or (b) or (c).

THEOREM 3. *Let $CV_\infty(X, E)$ be a Nachbin space. Then $VM_b(X, E')$ is linearly isomorphic to $CV_\infty(X, E)'$.*

4. **Bishop's generalized Stone-Weierstrass theorem [7].** If A is a subalgebra of $C(X)$, a subset $K \subset X$ is said to be *antisymmetric* for A if, for $f \in A$, the restriction $f|K$ being real-valued implies that $f|K$ is constant. Every antisymmetric set for A is contained in a maximal one, and the collection $\mathcal{K}_A$ of maximal antisymmetric sets for A forms a closed, pairwise disjoint covering of X (Glicksberg [3]). The following form of Bishop's generalized Stone-Weierstrass theorem is valid for Nachbin spaces (X is as in §3).

THEOREM 4. *Let $V \subset C^+(X)$ and let A be a subalgebra of $C(X)$ such that every $g \in A$ is bounded on the support of every $v \in V$. Let W be a vector subspace of $CV_\infty(X, E)$ which is an A-module. Then $f \in CV_\infty(X, E)$ is in the closure of W if and only if $f|K$ is in the closure of $W|K$ in $CV_\infty(K, E)$ for each $K \in \mathcal{K}_A$.*

If E is $\mathbf{R}$ or $\mathbf{C}$ the hypothesis $V \subset C^+(X)$ can be strengthened to $V \leqq C^+(X)$. If A is selfadjoint, the conclusion of Theorem 4 is that W is localizable under A in $CV_\infty(X)$ (see Definition 4, Nachbin [5]). Let $CV_\infty(X, E)$ be an A-module, where A satisfies the hypothesis of Theorem 4 and its maximal antisymmetric sets are sets reduced to a point, (e.g., $C_b(X)$, the algebra of all bounded continuous complex-valued functions). Under this hypothesis the following spectral synthesis result holds.

THEOREM 5. *Every proper closed A-submodule $W \subset CV_\infty(X, E)$ is contained in some closed A-submodule of codimension one in $CV_\infty(X, E)$ and is the intersection of all proper closed A-submodules of codimension one in $CV_\infty(X, E)$ which contain it.*

5. **Dieudonné theorem for density in tensor products of Nachbin spaces [8].** Let X and Y be two completely regular Hausdorff spaces and V and W two directed families of weights on X and Y respectively. Let $V \times W$ denote the set of all functions $(x, y) \mapsto v(x)w(y)$ on $X \times Y$. Let A be a locally convex topological algebra and let E and F be two locally convex spaces which are topological modules over A. Then $E \otimes_A F$ is defined to be the quotient space $(E \otimes F)/D$, where $E \otimes F$ has the projective tensor product topology and D is the closed linear span of the elements of the form $au \otimes v - u \otimes av$, where $a \in A$, $u \in E$, $v \in F$. If $f \in CV_\infty(X, E)$ and $g \in CW_\infty(Y, F)$, then $f \otimes_A g$ belongs to $C(V \times W)_\infty(X \times Y, E \otimes_A F)$, where $f \otimes_A g$ denotes the map $(x, y) \mapsto f(x) \otimes_A g(y)$.

THEOREM 6. *The vector subspace of all finite sums of mappings of the form $f \otimes_A g$, where $f \in CV_\infty(X, E)$ and $g \in CW_\infty(Y, F)$, is dense in $C(V \times W)_\infty(X \times Y, E \otimes_A F)$.*

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