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LOCALIZATION AND COMPLETION IN HOMOTOPY THEORY[1]

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1. **Introduction.** For each *space* X (i.e. simplicial set with only one vertex) and *solid ring* R (i.e. commutative ring with 1, for which the multiplication map $R\otimes R\to R$ is an isomorphism [3]) we shall construct, in a *functorial* manner, a space $X_R^\wedge$, the *R-completion* of X, and discuss some of its properties. The proofs will be given elsewhere.

If $R\subset Q$ (i.e. R is a subring of the rationals) and $\pi_1 X=0$, then $\pi_* X_R^\wedge \approx \pi_* X\otimes R$ and $X_R^\wedge$ is a *localization* in the sense of [7], [9] and [11].

If $R=Z_p$ (the integers modulo a prime p), $\pi_1 X=0$ and $\pi_n X$ is finitely generated for all n, then $\pi_* X_R^\wedge$ is the usual p-profinite completion of $\pi_* X$ and $X_R^\wedge$ is a *p-completion* in the sense of [8] and [11].

This note is, in some sense, a continuation of [2]. However, our present construction is (although *homotopically equivalent to*) completely different from the one of [2] and has the advantage that it can easily be generalized to a *functorial* notion of *fibre-wise R-completion.* In [2] we used *cosimplicial* methods, while here the basic tool is that of

2. **The R-completion of a group.** To define this notion we call a group N an *R-nilpotent* group if N has a *central series*

$$1 = N_k \subset \cdots \subset N_j \subset \cdots \subset N_0 = N$$

such that for each j the quotient group N_j/N_{j+1} *admits an R-module structure.* The R-completion of a group G then is the group $G_R^\wedge$ obtained by combining Artin-Mazur [1, §3] with an inverse limit, i.e. by taking the inverse limit [1, p. 147] of the functor which assigns to every homomorphism $G\to N$, where N is R-nilpotent, the group N, and to every commutative triangle

$$\begin{array}{ccc} & \nearrow & N \\ G & & \downarrow \\ & \searrow & N' \end{array}$$

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with N and N' both R-nilpotent, the map $N \to N'$. Clearly this R-completion comes with a natural map $G \to G_R\hat{}$ and indeed *the functor* $(\ )_R\hat{}$ *has the structure of a triple on the category of groups.*

Some "well-known" special cases are:

I. If $R=Z$ (the integers), then $G_R\hat{} = \text{proj lim } G/\Gamma_i G$, where $\{\Gamma_i G\}$ is the *lower central series* [4].

II. If $R=Z_p$, then $G_R\hat{} = \text{proj lim } G/\Gamma_i^{(p)} G$, where $\{\Gamma_i^{(p)} G\}$ is the *p-lower central series* [4].

III. If $R=Z_p$ and G is finitely generated, then $G_R\hat{}$ is the *p-profinite completion* of Serre [10, I-5] and thus if G is also abelian, then $G_R\hat{} \approx G \otimes$ (the p-adic integers).

IV. If $R=Q$ and G is nilpotent (i.e. Z-nilpotent), then $G_R\hat{}$ is the *Mal'cev completion* [5], [9, p. 279].

V. If $R \subset Q$ and G is abelian, then $G_R\hat{} \approx G \otimes R$.

It is not hard to verify that there also is a notion of *relative R-completion,* which assigns to a short exact sequence of groups $1 \to F \to G \to H \to 1$ a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & F & \to & G & \to & H & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & F_R\hat{} & \to & G_R\hat{}' & \to & H & \to & 1 \end{array}$$

with the bottom row also exact. Here $G_R\hat{}'$ is the group with generators (g, h), where $g \in F_R\hat{}$, and relations

$$(gf, h) = (g, fh), \qquad f \in F,\ g \in G,\ h \in F_R\hat{},$$
$$(g, h)(g', h') = (gg', g'(h)h'), \quad g, g' \in G,\ h, h' \in F_R\hat{},$$

where g' also denotes the automorphism of $F_R\hat{}$, which is the R-completion of the automorphism of F, which, in turn, is the restriction of the inner automorphism of G induced by g'.

3. **The R-completion of a space.** Let X be a "space," i.e. a simplicial set with only one vertex. Then we define its *R-completion,* $X_R\hat{}$, by

$$X_R\hat{} = \overline{W}(GX)_R\hat{}$$

i.e. we apply to X the *loop group* functor G [6], then *R-complete dimensionwise* and finally apply the *classifying* functor W [6]. Clearly this R-completion also comes with a natural map

$$X \to \overline{W}GX \to \overline{W}(GX)_R\hat{} = X_R\hat{}$$

and *this functor* $(\ )_R\hat{}$ *has the structure of a triple on the category of "spaces."*

Using the relative R-completion one can, in the same way, obtain the *fibrewise R-completion* of a fibration of "spaces" $F \to E \to B$, i.e. a functorial commutative diagram

$$\begin{array}{ccccc} F & \to & E & \to & B \\ \downarrow & & \downarrow & & \downarrow = \\ F_R^\wedge & \to & E_R^{\wedge\prime} & \to & B \end{array}$$

for which the map $E_R^{\wedge\prime} \to B$ is a fibre map and the inclusion of $F_R^\wedge$ in its fibre is a homotopy equivalence.

Depending on how much $X_R^\wedge$ resembles X, one can consider three classes of spaces: *R-complete*, *R-good* and *R-bad* ones. First the

4. **R-complete spaces.** A space X will be called *R-complete* if the natural map $X \to X_R^\wedge$ induces an *isomorphism* $\pi_* X \approx \pi_* X_R^\wedge$. For example

4.1 *The space $K(B, n)$ is R-complete if B admits an R-module structure* $(n \geqq 1)$.

To obtain a wider class of R-complete spaces we will say that a group π *acts R-nilpotently* on a left π-module M, if M has a π-module filtration

$$1 = M_k \subset \cdots \subset M_j \subset \cdots \subset M_0 = M$$

such that each quotient M_j/M_{j+1} has trivial π-action and admits an *R-module* structure, and call a space X *R-nilpotent* if $\pi_1 X$ is R-nilpotent and $\pi_1 X$ acts R-nilpotently on $\pi_n X$ for $n \geqq 2$. This is equivalent to saying that *the Postnikov tower of X can be refined to a tower of principal fibrations where each fibre is a $K(B, n)$ with B an R-module.* Now we have

4.2 *Every R-nilpotent space is R-complete.*

This follows readily from 4.1 and the following lemma (with $Y = *$).

4.3 FIBRE SQUARE LEMMA. *Consider the two commutative squares of connected spaces*

$$\begin{array}{ccc} F & \to & X \\ \downarrow & & \downarrow \\ Y & \to & B \end{array} \qquad\qquad \begin{array}{ccc} F_R^\wedge & \to & X_R^\wedge \\ \downarrow & & \downarrow \\ Y_R^\wedge & \to & B_R^\wedge \end{array}$$

If the first one of these is a fibre square with simply connected lower right corner, then so is, up to homotopy, the second one.

5. R-**good spaces.** A space X will be called *R-good* if the natural map $X \to X_R\hat{}$ induces an *isomorphism* $H_*(X; R) \approx H_*(X_R\hat{}; R)$. Moreover one has

5.1 *A space X is R-good if and only if $X_R\hat{}$ is R-complete.*

This follows, using the triple structure of $(\ )_R\hat{}$, from

5.2 *A map $f: X \to Y$ induces an isomorphism $H_*(X; R) \approx H_*(Y; R)$ if and only if it induces an isomorphism $\pi_* X_R\hat{} \approx \pi_* Y_R\hat{}$.*

Another consequence of this is that for an R-good space X the natural map $i: X \to X_R\hat{}$ is, up to homotopy, characterized by each of the following universal properties:

(i) *For any map $f: X \to Y$ inducing an isomorphism $H_*(X; R) \approx H_*(Y; R)$ there is a unique homotopy class of maps $g: Y \to X_R\hat{}$ such that $gf \sim i$.*

(ii) *For any map $f: X \to Y$ where Y is R-complete, there is a unique homotopy class of maps $h: X_R\hat{} \to Y$ such that $hi \sim f$.*

It seems hard to say when a space is R-good. Some partial results in this direction are

5.3 *Let $R = Z_p$ or $R \subset Q$ and let B be an abelian group. Then $K(B, n)$ is R-good for all n.*

Combining this with 4.3 one gets

5.4 *Let $R = Z_p$ or $R \subset Q$. Then every Z-nilpotent space is R-good.*

This is not best possible. For instance one has

5.5 *Let $R = Z_p$ or $R \subset Q$ and let X be a space such that $H_1(X; R) = 0$. Then X is R-good.*

6. R-**bad spaces.** A space X is called *R-bad* if it is *not R-good.* It turns out that R-bad spaces are "very bad."

6.1 *If X is R-bad, then so is $X_R\hat{}$, i.e. no iterated R-completion of X is R-complete.*

An example of a Z-bad space is an infinite wedge of circles, but we do not know whether the wedge of two circles is Z-bad.

7. **Homotopy groups of an** R-**completion.** The following result illustrates the close relation between the R-completion functor for spaces and the one for groups.

7.1 *Let X be a Z-nilpotent space, and unless $R \subset Q$, suppose that $\pi_n X$ is finitely generated for all n. Then there is a natural isomorphism*

$$\pi_n(\hat{X_R}) \approx (\pi_n X)\hat{_R}, \qquad n \geqq 1.$$

This is proved by combining 4.3 with computations for $K(B, n)$'s. For more results in the case $R = Z_p$ see [2, 7.2].

We end with the following consequence of 4.3 and 7.1, which is useful in computing R-completions of groups:

7.2 *Let N be a nilpotent (i.e. Z-nilpotent) group, let $B \subset N$ be a normal subgroup and, unless $R \subset Q$, suppose that N is finitely generated. Then the following sequence is exact:*

$$1 \to \hat{B_R} \to \hat{N_R} \to (N/B)\hat{_R} \to 1.$$

References

1. M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Math., no. 100, Springer-Verlag, Berlin and New York, 1969. MR **39** #6883.

2. A. Bousfield and D. Kan, *Homotopy with respect to a ring*, Proc. Sympos. Pure Math., vol. 22, Amer. Math. Soc., Providence, R. I., 1971, pp. 59–64.

3. ———, *The core of a ring* (to appear).

4. E. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209.

5. A. Mal'cev, *Nilpotent groups without torsion*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 201–212. (Russian) MR **10**, 507.

6. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Math. Studies, no. 11, Van Nostrand, Princeton, N. J., 1967. MR **36** #5942.

7. M. Mimura, G. Nishida and H. Toda, *Localization of* CW-*complexes and its applications* (to appear).

8. D. Quillen, *An application of simplicial profinite groups*, Comment Math. Helv. **44** (1969), 45–60. MR **39** #3490.

9. ———, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR **41** #2678.

10. J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Math., no. 5, Springer-Verlag, Berlin and New York, 1964. MR **31** #4785.

11. D. Sullivan, *Geometric topology*. I: *Localization, periodicity and Galois symmetry*, M.I.T., Cambridge, Mass., 1970 (mimeographed notes).

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