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CONDITION (C) AND GEODESICS ON SOBOLEV MANIFOLDS[1]

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In this note, we shall present a method to establish condition (C) by Palais and Smale [7] (Theorem 1) and announce a particular result (Theorem 2), which e.g. applies immediately to obtain the existence of minimal geodesics on Sobolev manifolds (Theorem 3). The method is based on the notion of weak submanifolds, which allows the introduction of some classical concepts from functional analysis in infinite-dimensional intrinsic differential geometry.

We shall call a Banach manifold M a weak submanifold of another Banach manifold M_0, iff for any point x_0 in the closure of M in M_0 there is an open neighbourhood U_0 of x_0 in M_0 together with a chart $\phi_0: U_0 \to \phi_0(U_0) \subset \boldsymbol{E}_0$ for M_0 and a Banach space $\boldsymbol{E}$, which is a linear subspace of $\boldsymbol{E}_0$ with a continuous inclusion $\boldsymbol{E} \subset \boldsymbol{E}_0$, such that the restriction of ϕ_0 to $U = M \cap U_0$ is a chart $\phi: U \to \phi(U) = \phi_0(U_0) \cap \boldsymbol{E}$ for M. We call such a chart for M a weak chart (at x_0).

Now let M, M_0 be Banach manifolds with M a weak submanifold of M_0. We shall call a function f on M weakly proper with respect to M_0, iff any subset A of M, such that f is bounded on A, is relatively compact in M_0, Moreover, we shall call f locally bounding with respect to M_0, iff for any point x_0 in M_0 and constant K, there is a weak chart for M at x_0 and a constant B, such that $\|x\| < B$ for all x in $\phi(U)$ with $f(x) < K$. Finally, we shall call a C^1 function f on M locally coercive with respect to M_0, iff for any point x_0 in M_0 and constant B, there is a weak chart for M at x_0 and constants $\lambda > 0$ and C, such that

$$(1) \qquad (Df(y) - Df(x))(y - x) \geqq \lambda \|y - x\|^2 - C\|y - x\|_0^2$$

holds for all x, y in $\phi(U)$ with $\|x\| < B$ and $\|y\| < B$, or equivalently in case f is of class C^2, such that

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$$(2) \qquad D^2 f(x)(\xi, \xi) \geqq \lambda\|\xi\|^2 - C\|\xi\|_0^2$$

holds for all x in $\phi(U)$ with $\|x\| < B$ and all ξ in $\boldsymbol{E}$. Here $\| \ \|$, $\| \ \|_0$ denote the norms in $\boldsymbol{E}$, $\boldsymbol{E}_0$.

THEOREM 1. *Let M be a C^2 Banach manifold with a Finsler structure and let f be a C^1 function on M. Suppose there exists a C^2 Banach manifold M_0 containing M as a weak submanifold, such that f is locally coercive, locally bounding and weakly proper with respect to M_0. Then f satisfies condition* (C).

PROOF. Let x_n be a sequence in M, such that $f(x_n)$ is bounded and $\|df(x_n)\|$ converges to zero. Since f is weakly proper w.r. to M_0, we can choose a subsequence of x_n converging in M_0 and then choose a weak chart at the limit, such that the subsequence is norm bounded and satisfies inequality (1). Then the subsequence obviously converges in M.

REMARKS. Inequality (1) (and (2)) applies mainly in the case M is a Hilbert manifold. More generally, the theorem remains true, if inequality (1) is replaced by

$$(1') \qquad (Df(y) - Df(x))(y - x) \geqq \phi(\|y - x\|) - \psi(\|y - x\|_0),$$

where ϕ is a strictly monotone function on $\boldsymbol{R}^+$ with $\lim_{t\to 0} \phi(t) = 0$ and ψ is a function on $\boldsymbol{R}^+$ with $\lim_{t\to 0} \psi(t) = 0$. This inequality is similar to a condition used by Palais [6, Theorem 19.21] to establish condition (C) for a function on an imbedded Sobolev manifold. Also in the works of Palais [5], Smale [9], Saber [8], Uhlenbeck [10] and the author [3], [4], where condition (C) is established for concrete variation integrals, we find the properties introduced here. In [4] there is constructed a large class of variation integrals (energy integrals), which are locally coercive and locally bounding but possibly not weakly proper (this can also be observed in [10]). Thus it seems to be preferable to separate these properties in different concepts, although they are strongly related. However, these concepts should of course be considered in the state of development.

Let N be a compact and connected C^∞ Riemannian manifold of dimension n and possibly with boundary. Let $\pi: W \to N$ be a finite dimensional C^∞ fibre bundle with a RMC structure [4], such that the fibres of W are complete Riemannian manifolds without boundary. Suppose moreover that the fibres of W are compact, if the boundary of N is empty. $H^k(W)$, with an integer $k > n/2$, denotes the Sobolev manifold of sections of the Sobolev class $H^k = L_k^2$ in W [6], [2], [4]. If f is a section in W, then $T_f W = f^* T_2 W$ denotes the pullback by f of the vertical tangent bundle $T_2W\to W$. $H^k(T_fW)$ is then the tangent space of $H^k(W)$ at the point f in $H^k(W)$. We give $H^k(W)$ the Riemannian structure induced by the RMC structure of W and then $H^k(W)$ is a C^∞ complete Riemannian manifold [4]. The inner product of two tangent vectors ξ, η in $H^k(T_fW)$ is given by

$$\langle \xi, \eta\rangle_k = \sum_{i=0}^{k}\int_N \langle \nabla^i\xi, \nabla^i\eta\rangle,$$

where $\nabla^i\xi\in H^{k-i}(L^i(TN, T_fW))$ is the covariant derivative of ξ of order i [2], [4]. We have on $H^k(W)$ an integrable distribution, which assigns to each $f\in H^k(W)$ the closed subspace $H_0^k(T_fW)$ of $H^k(T_fW)$ (the closure of the linear subspace of sections in $H^k(T_fW)$ with compact support in the interior of N). We denote by $H^k(W)_h$ the maximal connected integral manifold of this distribution, which contains h. All sections in $H^k(W)_h$ agree with h on the boundary of N and take the same Dirichlet H^k boundary values as h. $H^k(W)_h$ is then a closed C^∞ Riemannian submanifold of $H^k(W)$.

Let $I=[0, 1]$ be the unit interval, h a C^∞ section in W and $u: I\to H^k(W)_h$ a C^∞ curve. Then $H^1(I, H^k(W)_h)_u$ is a C^∞ Riemannian manifold and consists of all H^1 curves $\alpha: I\to H^k(W)_h$ homotopic with u and with fixed endpoints. We have a C^∞ function E on $H^1(I, H^k(W)_h)_u$, the so-called energy function, defined by

$$E(\alpha) = \tfrac{1}{2}\int_0^1 \left\|\partial\alpha(t)\right\|_k^2\, dt.$$

The critical points of E are exactly the geodesics in $H^k(W)_h$ connecting $u(0)$ with $u(1)$ and homotopic with u.

THEOREM 2. *The energy function E on $H^1(I, H^k(W)_h)_u$ satisfies condition* (C).

THEOREM 3. *Any two points in the Riemannian manifold $H^k(W)_h$ can be joined by a geodesic of minimal length.*

In [1] Dowling proves a result analogous to Theorem 2, but we have found his treatment of the intrinsic metrics incomplete. In the proof of Lemma 5.4, he forgets that the map $\nabla_x^r\delta_t$ is nonlinear and in the proof of Lemma 5.5 it is not possible to finish "by a simple induction."

We shall now describe briefly how Theorem 1 can be applied to prove Theorem 2. We use the following properties of the Sobolev manifolds:

1. $H^k(W)_h$ is a weak submanifold of $C^0(W)_h$ and $H^1(I, H^k(W)_h)_u$ is a weak submanifold of $C^0(I, C^0(W)_h)_u$.

2. There is a constant $c>0$, such that $\|\xi\|_{C^0}\leq c\|\xi\|_k$, for $\xi\in H^k(T_fW)$,

$f \in H^k(W)$ [4].

3. The energy function $E_k(f) = \frac{1}{2}\|\nabla f\|_{k-1}^2$ on $H^k(W)_h$ is a C^∞ function, which is weakly proper, locally bounding and locally coercive w.r. to $C^0(W)_h$, in particular E_k satisfies condition (C) [4].

4. E_k is bounded on bounded subsets of $H^k(W)_h$.

5. The imbedding $H^k(W)_h \subset C^0(W)_h$ is compact, i.e. bounded subsets of $H^k(W)_h$ are relatively compact in $C^0(W)_h$ [4].

For any α in $H^1(I, H^k(W)_h)_u$ we have the following estimate on the distance d_k in $H^k(W)_h$:

$$d_k(\alpha(t), \alpha(s)) \leqq \left| \int_t^s \|\partial\alpha(t)\|_k \, dt \right| \leqq (|t - s|)^{1/2}(2E(\alpha))^{1/2}.$$

Thus if E is bounded on a subset A of $H^1(I, H^k(W)_h)_u$, then $A(I) = \{\alpha(t) \mid \alpha \in A,\ t \in I\}$ is a bounded subset of $H^k(W)_h$ and thus relatively compact subset of $C^0(W)_h$ (by 5). Moreover, A is an equicontinuous subset of $C^0(I, H^k(W)_h)_u$ and also of $C^0(I, C^0(W)_h)_u$ (by 2). Then A is a relatively compact subset of $C^0(I, C^0(W)_h)_u$ as a consequence of the Ascoli-Arzela Theorem, or E is weakly proper w.r. to $C^0(I, C^0(W)_h)_u$. Now E_k is bounded on $A(I)$ (by 4), then using that E_k is locally bounding w.r. to $C^0(W)_h$ (by 3) and a local formula for E, it is not difficult to show that E is locally bounding w.r. to $C^0(I, C^0(W)_h)_u$. It is also not difficult to show that E is locally coercive w.r. to $C^0(I, C^0(W)_h)_u$ by local estimates.

References

1. J. Dowling, *Finsler geometry on Sobolev manifolds*, Proc. Sympos. Pure Math., vol. 15, Amer. Math. Soc., Providence, R. I., 1970, pp. 1–10. MR 41 #7724.

2. H. I. Elíasson, *Geometry of manifolds of maps*, J. Differential Geometry 1 (1967), 169–194. MR 37 #2268.

3. ———, *Morse theory for closed curves*, Sympos. Infinite Dimensional Topology, Louisiana State University, Baton Rouge, La., 1967.

4. ———, *Variation integrals in fibre bundles*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I., 1970, pp. 67–89.

5. R. S. Palais, *Morse theory on Hilbert manifolds*, Topology 2 (1963), 299–340. MR 28 #1633.

6. ———, *Foundations of global non-linear analysis*, Benjamin, New York, 1968. MR 40 #2130.

7. R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. 70 (1964), 165–172. MR 28 #1634.

8. J. C. Saber, *Manifolds of maps*, Dissertation, Brandeis University, Waltham, Mass., 1965.

9. S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) 80 (1964), 382–396. MR 29 #2820.

10. K. K. Uhlenbeck, *The calculus of variations and global analysis*, Dissertation, Brandeis University, Waltham, Mass., 1968.

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