BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

ON THE EXISTENCE OF A CONTROL MEASURE FOR STRONGLY BOUNDED VECTOR MEASURES

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Communicated by Bertram Yood, April 29, 1971

In this note we extend a theorem of Bartle, Dunford, and Schwartz [1] which states that for every countably additive measure defined on a σ-algebra there exists a positive "control measure" ν such that $\nu(E)\to 0$ if and only if $\|\mu\|(E)\to 0$, where $\|\mu\|$ is the semivariation of μ. In this paper, μ, which is defined on a ring Σ, is assumed to be finitely additive and strongly bounded (s-bounded) [8] (that is $\mu(E_i)\to 0$ whenever $\{E_i\}$ is a disjoint sequence of sets). The existing decomposition and extension theorems for vector measures can now be easily deduced by using the control measure. These applications will be presented in [2].

$\mathfrak{X}$ is a Banach space over the reals (the complex case is treated in a similar fashion); S^* is the unit sphere in the conjugate space of $\mathfrak{X}$. $\sigma(\mathcal{E})$ denotes the σ-algebra generated by the class of sets $\mathcal{E}$. A δ-ring is a ring of sets closed under countable intersections.

THEOREM 1. *Let Σ be a ring of subsets of a set S. $\mu:\Sigma\to\mathfrak{X}$ is finitely additive and s-bounded if and only if there exists a positive finitely additive bounded set function ν defined on Σ such that*

$$\lim_{\nu(E)\to 0} \mu(E) = 0$$

and

$$\nu(E) \leqq \sup\{\|\mu(F)\| : F \subseteq E,\ F \in \Sigma\}, \qquad E \in \Sigma.$$

SKETCH OF THE PROOF. First assume Σ is an algebra. Let T be the isometric isomorphism of $ba(S, \Sigma)$ onto $ba(S_1, \Sigma_1)$, where Σ_1 is the Stone algebra of all open-closed subsets of the compact totally disconnected Hausdorff space S_1 [4, IV.9]. U is the isometric isomorphism between $ba(S_1, \Sigma_1)$ and $ca(S_1, \Sigma_2)$, where $\Sigma_2=\sigma(\Sigma_1)$.

We prove that $\{(UT)(x^*\mu) : x^*\in S^*\}$ is uniformly countably additive on Σ_2. It suffices to show that $\{(UT)[(x^*\mu)^+] : x^*\in S^*\}$ is uniformly countably additive, where $x^*\mu=(x^*\mu)^+-(x^*\mu)^-$ is the Jordan

AMS 1970 *subject classifications*. Primary 22A45.

Key words and phrases. Stone algebra, unconditional convergence, weak convergence, weakly compact.

[1] This research was supported in partly by NSF Grant GP 28617.

decomposition of $x^*\mu$. Assume the contrary. There exist $A_i \in \Sigma_2$, $A_i \searrow \varnothing$, $\epsilon > 0$, $x_i^* \in S^*$ such that $|\lambda_i(A_i)| > \epsilon$, $i = 1, 2, \cdots$, where $\lambda_i = (UT)\,(x_i^*\mu)^+$. Let Σ_{01} be a countable subalgebra of Σ_1 such that all the A_i belong to $\Sigma_{02} = \sigma(\Sigma_{01})$. By a diagonal process, we may assume that the λ_i converge pointwise on Σ_{01}. Let λ denote this limit. By using the unconditional convergence property of s-bounded measures [8, 2.3], the Orlicz-Pettis theorem, and a result of Procelli [7], we can show that the sequence λ_i converges weakly to λ in $ba(S_1, \Sigma_{01})$. By a result of Leader [5] this implies that the λ_i are uniformly absolutely continuous with respect to a positive finitely additive bounded set function φ defined on Σ_{01}. The extensions of the λ_i to Σ_{02} are uniformly absolutely continuous with respect to the extension of φ to Σ_{02}. This yields a contradiction since the $A_i \in \Sigma_{02}$. Now since the $\{(UT)\,(x^*\mu)^+ : x^* \in S^*\}$ are uniformly countably additive and uniformly bounded on Σ_2, the Bartle-Dunford-Schwartz theorem yields a ν'_+ defined on Σ_2 that acts as a control measure for the family $\{(UT)\,(x^*\mu)^+ : x^* \in S^*\}$. ν'_- is obtained in a similar fashion. $1/2[(UT)^{-1}(\nu'_+ + \nu'_-)]$ yields the desired set function.

Now assume Σ is a ring. By analysing the structure of the algebra Σ^* generated by Σ, one can prove that μ can be extended to Σ^* and that the extension is s-bounded on Σ^*. This then reduces to the previous case.

REMARK 1. By means of an example found in [3], one can show that the conclusion of the theorem is false if the assumption of s-boundedness is dropped.

In view of the above theorem and the results in [9], we have the following:

COROLLARY. *Let $\mu : \Sigma \to \mathfrak{X}$ be finitely additive, where Σ is a ring. μ is s-bounded if and only if the range of μ is conditionally weakly compact.*

THEOREM 2. *Let $\mu : \Sigma \to \mathfrak{X}$ be countably additive, where Σ is a ring. Then there exists a countably additive bounded set function ν defined on Σ which is a control measure for μ if and only if μ is s-bounded.*

SKETCH OF PROOF. If μ is s-bounded, then by Theorem 1 there exists a bounded finitely additive control measure ν. The conclusion in Theorem 1 implies that $\nu(E) \leqq \|\mu\|(E) = \sup\{\|\mu(F)\| : F \subseteq E\}$. One can show that since μ is countably additive, $E_i \searrow \varnothing$ implies that $\|\mu\|(E_i) \to 0$; this in turn implies that ν is countably additive.

REMARK 2. Due to a recent result of Rybakov, one can choose the control measures appearing in Theorems 1 and 2 to be the total variation function of the measure $x^*\mu$, for some $x^* \in \mathfrak{X}^*$.

REMARK 3. Using Zorn's lemma, the control measures can be chosen to be dominated by $\|\mu\|$ and also be maximal with respect to the usual order defined on set functions.

REMARK 4. For the bounded case this answers the question posed by Dinculeanu and Kluvanek [3, p. 505] concerning the existence of control measures for vector measures defined on δ-rings. If one requires that ν be only finite valued and not bounded, then S. Ohba has shown [6] that ν exists, without the requirement of s-boundedness of μ, when $\mathfrak{X}$ is separable. Examples show that without the separability condition the result is false.

We say that a family Γ of vector measures is uniformly s-bounded if $\mu(E_i)\to 0$ uniformly for $\mu\in\Gamma$, whenever $\{E_i\}$ is a disjoint sequence of sets. This concept is equivalent to uniform countable additivity when the measures are countably additive on a σ-ring. The technique in the proof of Theorem 1 is used to prove the following theorem.

THEOREM 3. *Let Σ^* be the σ-ring generated by the ring Σ. If a uniformly bounded family of countably additive vector measures is uniformly s-bounded on Σ, then the family is uniformly countably additive on Σ^*.*

The following theorem contains a converse to the Nikodym theorem. The proof uses the above theorem and Lemma IV. 8.8 in [4].

THEOREM 4. *Let $\{\mu_n\}$ be a uniformly bounded sequence of countably additive vector measures defined on a σ-ring Σ^* generated by the ring Σ such that for every E in Σ $\lim_n \mu_n(E)$ exists. Then $\lim_n \mu_n(E^*)$ exists for every E^* in Σ^* if and only if $\{\mu_n\}$ is uniformly s-bounded on Σ.*

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