

INVARIANTS FOR SEMIFREE S^1 -ACTIONS

BY KATSUO KAWAKUBO¹

Communicated by Emery Thomas, June 9, 1971

1. Introduction. In [1], Atiyah and Singer obtained an invariant for certain S^1 -actions and in [5], Browder and Petrie used the invariant to distinguish certain semifree S^1 -actions so that they showed the following result.

THEOREM 0. *For any odd $n \geq 5$, $n = 2k - 1$, there are an infinite number of distinct semifree S^1 -actions on the Brieskorn $2n + 1$ spheres with fixed point set of codimension $4m$, any $m \neq k/2$, $m < (n - 1)/2$.*

The purpose of the present paper is to define some invariants for certain semifree S^1 -actions which are different from that of Atiyah and Singer (see Theorem 1). As an application, we can prove the above theorem of Browder and Petrie without the assumption $m \neq k/2$ (see Corollary 2). Our method is different from that of Atiyah and Singer, and Browder and Petrie. We use the Chern classes due to Borel and Hirzebruch [2] and Grothendieck (see [3]) and the bordism theory. The author wishes to thank Professor F. Uchida who kindly enlightened him about the structure of the normal bundle of fixed point set.

Detailed proof will appear elsewhere.

2. Definitions and statement of results.

DEFINITIONS. An action (M, ϕ, G) is called semifree if G acts freely outside the fixed point set.

Denote by $B(\xi)$, $S(\xi)$ and $CP(\xi)$, the total space of the disk bundle, the total space of the sphere bundle and the total space of the projective space bundle respectively associated to a complex vector bundle ξ . We remark the following. Given a semifree S^1 -action (M, ϕ, S^1) where M may have a boundary, the normal vector bundle of the fixed point set has the unique complex vector bundle structure such that the induced action of S^1 is the scalar multiplication when we regard S^1 as $\{z \mid z \in \mathbf{C}, |z| = 1\}$. By "complex bundle" we mean that such a complex structure is taken. Let (M, ϕ, S^1) be a semifree S^1 -action

AMS 1970 subject classifications. Primary 57D20, 57D85, 57E15.

Key words and phrases. Semifree S^1 -actions, Chern class, bordism theory, index, Stiefel-Whitney number, Pontrjagin number.

¹The author was partially supported by National Science Foundation grant GP7952X2.

where M is a closed manifold. We suppose that

- (i) The fixed point set $F(S^1, M^n)$ is a homology sphere;
- (ii) (M^n, ϕ, S^1) extends to a semifree S^1 -action (W^{n+1}, Φ, S^1) ($\partial W^{n+1} = M^n$ as S^1 -manifold) such that the fixed point set $F(S^1, W)$ is connected and the normal complex bundle of the fixed point set has the trivial Chern classes.

First we define Stiefel-Whitney numbers $W_I[F(S^1, W)]$ as follows where I denotes a partition of $\dim F(S^1, W)$. Let $W_i(F(S^1, W))$ be the i th Stiefel-Whitney class of $F(S^1, W)$. Since $\partial F(S^1, W)$ ($= F(S^1, M)$) is a homology sphere, the homomorphism

$$j^*: H^i(F(S^1, W), \partial F(S^1, W); \mathbf{Z}_2) \rightarrow H^i(F(S^1, W); \mathbf{Z}_2)$$

is an isomorphism for $0 < i \leq n - 2k$.

DEFINITION. For a nontrivial partition $I = (i_1, \dots, i_l)$ of $\dim F(S^1, W)$, the Stiefel-Whitney number $W_I[F(S^1, W)]$ is defined by the number

$$\langle j^{*-1}W_{i_1}(F(S^1, W)) \cdots j^{*-1}W_{i_l}(F(S^1, W)), [F(S^1, W), \partial F(S^1, W)] \rangle \in \mathbf{Z}_2,$$

where nontrivial means $I \neq \dim F(S^1, W)$ and $\langle \ , \ \rangle$ denotes the Kronecker index. For the trivial partition $I = \dim F(S^1, W)$, $W_I[F(S^1, W)]$ is defined by the number

$$\chi(F(S^1, W)) + 1 \pmod{2},$$

where $\chi(F(S^1, W))$ denotes the Euler number of $F(S^1, W)$.

When $\dim F(S^1, W) = n + 1 - 2k \equiv 0 \pmod{4}$, we define Pontrjagin numbers and the index of $F(S^1, W)$ as follows. Let $P_i(F(S^1, W))$ be the i th Pontrjagin class of $F(S^1, W)$. Since $\partial F(S^1, W)$ ($= F(S^1, M)$) is a homology sphere, the homomorphism

$$j^*: H^i(F(S^1, W), \partial F(S^1, W); \mathbf{Z}) \rightarrow H^i(F(S^1, W); \mathbf{Z})$$

is an isomorphism for $0 < i \leq n - 2k$.

DEFINITION. For a nontrivial partition $I' = (i_1, \dots, i_s)$ of $\dim F(S^1, W)/4 = (n + 1 - 2k)/4$, the Pontrjagin number $P_{I'}[F(S^1, W)]$ of $F(S^1, W)$ is defined by the number

$$\langle j^{*-1}P_{i_1}(F(S^1, W)) \cdots j^{*-1}P_{i_s}(F(S^1, W)), [F(S^1, W), \partial F(S^1, W)] \rangle \in \mathbf{Z}.$$

DEFINITION. The index $\tau(F(S^1, W))$ of $F(S^1, W)$ is defined by the signature of the cup product pairing

$$\begin{aligned} H^v(F(S^1, W), \partial F(S^1, W); \mathbf{R}) \otimes H^v(F(S^1, W), \partial F(S^1, W); \mathbf{R}) \\ \rightarrow H^{2v}(F(S^1, W), \partial F(S^1, W); \mathbf{R}) \end{aligned}$$

where $v = \dim F(S^1, W)/2 = (n + 1 - 2k)/2$.

Then we shall have

THEOREM 1. *Stiefel-Whitney numbers $W_I[F(S^1, W)]$ do not depend on the choice of such an extension (W, Φ, S^1) of (M, ϕ, S^1) . If $\dim F(S^1, W) \equiv 0 \pmod{4}$, Pontrjagin numbers $P_I[F(S^1, W)]$ and the index $\tau(F(S^1, W))$ do not depend on the choice of such an extension (W, Φ, S^1) of (M, ϕ, S^1) .*

As an application, we shall have

COROLLARY 2. *For any odd, $n \geq 5$, $n = 2k - 1$, there are an infinite number of distinct semifree S^1 -actions on the Brieskorn $(2n + 1)$ -spheres with fixed point set of codimension $4m$, any $m < (n - 1)/2$.*

3. Outline of the proof of Theorem 1. Suppose a semifree S^1 -action (M, ϕ, S^1) satisfies (i) and (ii) above. Let (W_1, Φ_1, S^1) and (W_2, Φ_2, S^1) be two extensions of (M, ϕ, S^1) satisfying the condition (ii). Denote by ξ_1 and ξ_2 the normal complex bundles of the fixed point sets $F(S^1, W_1)$ and $F(S^1, W_2)$ respectively. By combining the two actions, we have the action $(W, \Phi, S^1) = (W_1 \cup_{\text{id}} (-W_2), \Phi_1 \cup \Phi_2, S^1)$ where $-W_2$ is W_2 with the opposite orientation. It follows from the uniqueness of the complex structure that the normal bundle of the fixed point set $F = F(S^1, W_1) \cup (-F(S^1, W_2))$ of the action (W, Φ, S^1) has the complex vector bundle structure ξ whose restrictions to $F(S^1, W_1)$ and $F(S^1, W_2)$ are isomorphic to ξ_1 and ξ_2 respectively as complex vector bundles. Let $i_1: F(S^1, W_1) \rightarrow F$ and $i_2: F(S^1, W_2) \rightarrow F$ be the inclusions, then the i th Chern class $c_i(\xi)$ satisfies $i_1^* c_i(\xi) = c_i(\xi_1)$ and $i_2^* c_i(\xi) = c_i(\xi_2)$ and the Mayer-Vietoris exact sequence

$$\begin{aligned} \rightarrow H^{i-1}(F(S^1, M)) \xrightarrow{\delta} H^i(F) \xrightarrow{i_1^* \oplus i_2^*} \\ H^i(F(S^1, W_1)) \oplus H^i(F(S^1, W_2)) \rightarrow H^i(F(S^1, M)) \rightarrow \end{aligned}$$

shows that the homomorphism

$$i_1^* \oplus i_2^*: H^i(F) \rightarrow H^i(F(S^1, W_1)) \oplus H^i(F(S^1, W_2))$$

is an isomorphism for $0 < i \leq n - 2k$. Hence $c_i(\xi) = (i_1^* \oplus i_2^*)^{-1} \cdot (c_i(\xi_1) \oplus c_i(\xi_2)) = 0$ for $0 < i < (n + 1 - 2k)/2$. Next we show that $c_i(\xi)$ is zero also for $i \geq (n + 1 - 2k)/2$. Since it is trivial to prove in the case where $n + 1 - 2k$ is odd, we assume that $n + 1 - 2k$ is even. Let $t \in H^2(\mathbf{CP}(\xi))$ be the first Chern class of the canonical line bundle over $\mathbf{CP}(\xi)$. Then by Borel and Hirzebruch [2] and Grothendieck (see [3]), $c_{(n+1-2k)/2(\xi)}$ satisfies the following relation

$$i^{(n-1)/2} + p^*c_{(n+1-2k)/2}(\xi) \cdot i^{k-1} = 0$$

where $p: \mathbf{CP}(\xi) \rightarrow F$ is the projection map.

Since the classifying map $f: \mathbf{CP}(\xi) \rightarrow \mathbf{CP}^\infty$ of the bundle $S^1 \rightarrow S(\xi) \rightarrow \mathbf{CP}(\xi)$ represents the zero element of $\Omega_{n-1}(\mathbf{CP}^\infty)$, we have $i^{(n-1)/2} = 0$. It follows from the cohomology ring structure of $\mathbf{CP}(\xi)$ that $c_{(n+1-2k)/2}(\xi) = 0$. Thus all the Chern classes vanish.

Let $h: F \rightarrow BU(k)$ be the classifying map of the bundle ξ and let $*$: $F \rightarrow BU(k)$ be the constant map. Then (F, h) and $(F, *)$ represent the same element of $\Omega_{n+1-2k}(BU(k))$ by Conner-Floyd [6], i.e., there exist a manifold \bar{F} and a map $H: \bar{F} \rightarrow BU(k)$ such that $\partial\bar{F} = F \cup_{\text{disj}} (-F)$ and $H|_{\partial\bar{F}} = h \cup *$ where \cup_{disj} means disjoint union. Denote by $\bar{\xi}$ the induced bundle $H^1(\gamma(k))$ where $\gamma(k)$ is the universal bundle of the unitary group $U(\xi)$. Now we construct an oriented closed manifold

$$W' = (W - \text{Int } B(\xi)) \cup S(\bar{\xi}) \cup F \times D^{2k}$$

where the attaching maps are the obvious ones, since $\bar{\xi}|_{\partial\bar{F}} = \xi \cup_{\text{disj}} F \times \mathbb{C}^k$. Naturally we can define an action (W', Φ', S^1) with fixed point set F whose normal complex bundle is trivial. Hence Theorem 1 will follow from the following two lemmas.

LEMMA 3. *Let (M, ϕ, S^1) be a semifree S^1 -action on an oriented closed manifold M with connected fixed point set F^l . If the normal complex bundle ξ of the fixed point set F is trivial, then F represents the zero element of the oriented cobordism group Ω_l .*

LEMMA 4. *Let F_1^l, F_2^l be two oriented compact manifolds such that $\partial F_1 = \partial F_2$ and ∂F_1 is a homology sphere. By attaching the two manifolds along their boundaries, we obtain an oriented closed manifold $F = F_1 \cup (-F_2)$. Suppose that F represents the zero element of Ω_l . Then we have*

$$W_I[F_1] = W_I[F_2]$$

for each partition I of l . When $l \equiv 0 \pmod{4}$, we have

$$P_I[F_1] = P_I[F_2]$$

for each nontrivial partition I' of $l/4$, and

$$\tau(F_1) = \tau(F_2).$$

4. **Outline of the proof of Corollary 2.** By making use of the Brieskorn and Hirzebruch examples [4], [7], we can construct an infinite number of distinct semifree S^1 -actions, which are detected by the invariant τ of Theorem 1.

BIBLIOGRAPHY

1. M. F. Atiyah and I. M. Singer, *The index of elliptic operators*. III, Ann. of Math. (2) **87** (1968), 546–604. MR **38** #5245.
2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. **80** (1958), 458–538. MR **21** #1586.
3. R. Bott, *Notes on K-theory*, Harvard University, Cambridge, Mass., 1962.
4. E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14. MR **34** #6788.
5. W. Browder and T. Petrie, *Semi-free and quasi-free S^1 -actions on homotopy spheres*, Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), Springer, New York, 1970, pp. 136–146. MR **41** #7716.
6. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Band 33, Academic Press, New York; Springer-Verlag, Berlin, 1964. MR **31** #750.
7. F. Hirzebruch, “Singularities and exotic spheres,” *Séminaire Bourbaki*, 1966/67, Exposé 314, Benjamin, New York, 1968. MR **37** #10.

OSAKA UNIVERSITY, OSAKA, JAPAN

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540