BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

AN EXTENSION OF KHINTCHINE'S ESTIMATE FOR LARGE DEVIATIONS TO A CLASS OF MARKOV CHAINS CONVERGING TO A SINGULAR DIFFUSION[1]

BY H. BREZIS, W. ROSENKRANTZ AND B. SINGER

Communicated by Harry Kesten, June 9, 1971

1. For $S(n)$ a sum of n independent identically distributed random variables with mean zero and variance one, Khintchine obtained the estimate

$$P(S(n)/\sqrt{n} \geqq \alpha_n) = \exp\left(-\tfrac{1}{2}\alpha_n^2(1+o(1))\right) \tag{1.1}$$

where $\alpha_n \uparrow +\infty$ with a certain rate of growth. Recently an elementary proof of this estimate was given by Mark Pinsky in [4]. Using Pinsky's method and some nontrivial estimates in the theory of partial differential equations we prove that (1.1) holds for an interesting class of Markov chains converging to a singular diffusion process on the half line $\overline{R}_+ = [0, \infty]$. The random walks we shall study have state space $I^+ = \{0, 1, 2, \cdots\}$ and transition probabilities $p(i, j)$ given by $p(i, i) = 0$, $i = 0, 1, 2, \cdots$,

$$\begin{aligned} p(i, i+1) &= \tfrac{1}{2}(1+\gamma/i), \\ p(i, i-1) &= 1 - p(i, i+1), \\ p(0, 1) &= 1, \quad \text{the “reflecting barrier condition” at the origin.} \end{aligned} \tag{1.2}$$

In addition we assume $0 \leqq \gamma < 1$.

$\{X(n): n = 0, 1, \cdots,\}$ denotes the random walk with transition matrix $p(i,j)$ defined by (1.2), and $P_x(\ )$ denotes the measure induced on sequences of nonnegative integers by $\{X(n): X(0) = x\}$.

THEOREM 1.1. *If $\{\alpha_n\}$ is any sequence increasing to $+\infty$ satisfying the condition $\lim_{n\to\infty} \alpha_n^2 - (\log n)/2 = -\infty$, then for each $\epsilon > 0$ there exists an integer $N(\epsilon)$ so that for $n \geqq N(\epsilon)$ we have*

$$\exp\left(-\tfrac{1}{2}\alpha_n^2(1+\epsilon)\right) \leqq P_0(X(n)/\sqrt{n} \geqq \alpha_n) \leqq \exp\left(-\tfrac{1}{2}\alpha_n^2(1-\epsilon)\right). \tag{1.3}$$

AMS 1970 *subject classifications*. Primary 60F10; Secondary 60J10, 60J25.

[1] The research in this paper was supported in part by the National Science Foundation under grant GP-9640 at Columbia University and by the National Science Foundation under grant GP-11460 at the Courant Institute. Reproduction in whole or in part is permitted for any purpose of the United States Government.

It is to be observed that such estimates were first obtained in very special cases and using completely different techniques by Rosenkrantz [5]. The estimate (1.3) includes as well the class of random walks studied in [5].

We sketch the proof, as the details will be published elsewhere. To the random walk $\{X(n): n=0, 1, \cdots\}$ we can associate a sequence of stochastic processes $\{X_n(t)\}$ first studied by Lamperti [3]. Set $X_n(t)=X([nt])/\sqrt{n}$ where $[nt]$ denotes the integer part of nt. Lamperti, op.cit., showed that $\lim_{n\to\infty} X_n(t)=Y(t)$ in the sense of weak convergence of stochastic processes where $Y(t)$ is a Markov process on R_+ with infinitesimal generator G given by

$$Gf(x) = \tfrac{1}{2}f''(x) + \gamma f'(x)/x, \qquad x \geqq 0. \tag{1.4}$$

That is, let $U(x, t)=E_x f(Y(t))$. Then, provided f is smooth enough, $U(x, t)$ satisfies the Kolmogorov backward differential equation

$$\begin{aligned} U_t(x, t) &= GU(x, t) = \tfrac{1}{2}U_{xx}(x, t) + \gamma U_x(x, t)/x, \\ U(x, 0) &= f(x), \qquad U_x(0, t) = 0. \end{aligned} \tag{1.5}$$

In a previous work [1] we have characterized the class of functions f for which smooth solutions to the singular parabolic partial differential equation (1.5) exist. These regularity theorems are essential in deriving the estimate (1.7) below.

We then construct a sequence of random walks $\{Y_n(t)\}$ with the following two properties:

$$P_{[\sqrt{n}\cdot a]}(X_n(t) \geqq \alpha) \leqq P_{[\sqrt{n}\cdot a]}(Y_n(t) \geqq \alpha), \qquad a \geqq 0, \alpha \geqq 0, n \geqq 1. \tag{1.6}$$

$$| E_x f(Y_n(t)) - E_x f(Y(t)) | = O(n^{-\sigma}), \qquad \sigma > 0, \tag{1.7}$$

where f is a smooth function. Because of (1.6) we refer to the $Y_n(t)$ process as the "dominating random walk."

In the case of i.i.d. random variables the estimate (1.7) is (cf. [4]) trivial because the operation of differentiation commutes with the infinitesimal generator of the Brownian motion semigroup. This is certainly not the case for the generator G. Our most difficult step then is to derive the rate of convergence estimate (1.7). Then Pinsky's idea, op. cit., can be exploited to yield (1.3) for the sequence of random variables $Y_n(1)$. From the "domination inequality" (1.6) and a lower bound in [5] estimate (1.3) is readily inferred.

For the purposes of the law of the iterated logarithm, (1.3) suffices. Nevertheless an exact asymptotic estimate of the middle term of (1.3) is not without interest.

THEOREM 1.2. *If $\{\alpha_n\}$ is any sequence increasing to $+\infty$ and satis-*

fying the condition $\lim_{n\to\infty} \alpha_n^2 - (\log n)/8 = -\infty$ *then*

$$P_0(X(n)/\sqrt{n} > \alpha_n) \sim P_0(Y(1) > \alpha_n).$$

Finally we use our results to derive a law of the iterated logarithm for the random walk $X(n)$. That is we prove

$$P_0\left(\limsup_{n\to\infty} \frac{X(n)}{(2n \log \log n)^{1/2}} = 1\right) \leqq 1. \tag{1.8}$$

Thus we have obtained a genuine extension of the law of the iterated logarithm for a class of Markov chains converging to a singular diffusion.

For a general account of the ideas used herein we refer the reader to Trotter [6]. Mention should also be made of Khintchine's proof of the DeMoivre-LaPlace limit theorem, for which see Itô-McKean [**2**, pp. 10–11].

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Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

Columbia University, New York, New York 10027

Current address: (H. Brezis) Faculté des Sciences, 9 Quai Saint-Bernard, Paris 5, France

Current address: (W. Rosenkrantz) University of Massachusetts, Amherst, Massachusetts 01002