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FOURIER TRANSFORMS OF THE BEURLING CLASSES $\mathfrak{D}_\omega$, $\mathcal{E}'_\omega$[1]

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0. It is well known (cf. [6], [7], [8]) that the topological vector spaces $\mathfrak{D}$ and $\mathcal{E}'$ (of test functions and distributions with compact support, respectively) exhibit several surprising similarities in their topological character; and yet the definitions of these spaces do not indicate why this should be so. The situation changes if we consider their Fourier images $\hat{\mathfrak{D}}$ and $\hat{\mathcal{E}}'$. Thus, for instance, L. Ehrenpreis noticed [7, pp. 161–163] that, outside a certain neighborhood of the real subspace in C^n, "the topologies of $\hat{\mathfrak{D}}$ and $\hat{\mathcal{E}}'$ are the same." Ehrenpreis also showed that this property is important in the study of hypoellipticity and other questions (cf. [6, pp. 63–65]). The main objective of this note is to provide a simple explanation of the relationship between the spaces $\mathfrak{D}$ and $\mathcal{E}'$. Our approach, which is also based on the Fourier transform, can be briefly described as follows: We shall construct functions $k(\zeta)$, $\zeta=\xi+i\eta\in C^n$, of the form $k(\zeta) = \sum_{s=-\infty}^{\infty} e_s(\zeta)$—where $e_s(\zeta)$ are certain exponentials, and the index s corresponds to the order of differentiation in $\mathcal{E}'$—such that (i) if $\mathcal{K}$ denotes the family of all such functions k, then the space $\hat{\mathcal{E}}'$ can be characterized, both algebraically and topologically, in terms of majoration of its elements by the functions in $\mathcal{K}$; (ii) moreover, if instead of considering the complete "Laurent" series $k=\sum_{s=-\infty}^{\infty}\cdots$, we take only their "Taylor" parts, i.e., $k^+=\sum_{s=0}^{\infty}\cdots$, then the family $\mathcal{K}^+=\{k^+\}_{k\in\mathcal{K}}$ defines similarly the space $\hat{\mathfrak{D}}$. From here it is easy to deduce the above mentioned observation of Ehrenpreis as well as some other facts. However, in the present note we state explicitly only what seems to be a new description of the convex hull of singular support of a distribution. Finally it should be mentioned that we shall discuss most of these questions in the frame of a more general distribution theory due to A. Beurling (cf. [4]). In doing so we shall not only gain in simplicity and generality, but some new information about the Beurling classes $\mathcal{E}_\omega$, $\mathcal{E}'_\omega$ will be obtained as well. As the proofs are rather technical, they will be published elsewhere

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(cf. [2], [3], where some further applications are also given). Theorems 1 and 2 below answer the questions formulated in the end of [5].

1. If ω is a nonnegative subadditive function defined on R^n and satisfying conditions (α), (β), (γ) of [4], then by regularizing ω if necessary, we can always assume ω to be smooth and $\geqq 1$. Let $\{K_s\}$ be any strictly increasing sequence (i.e. $K_s \subset \text{int } K_{s+1}$) of compact sets, $\bigcup_{s \geqq 1} K_s = R^n$. The space $\mathfrak{D}_\omega$ is defined as $\mathfrak{D}_\omega = \lim \text{ind}_{s\to\infty} \mathfrak{D}_\omega(K_s)$ where each

$$\mathfrak{D}_\omega(K_s) = \{\phi \in L^1(R^n) : \text{supp } \phi \subset K_s \text{ and}$$
$$\|\phi\|_\lambda = \int |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty \ (\forall \lambda > 0)\}.$$

(The Fourier transform of a function $\phi \in L^1(R^n)$ is defined here by $\hat{\phi}(\xi) = \int \phi(x) e^{-ix\cdot\xi} dx$.) The space $\mathcal{E}_\omega$ is the set of all functions ϕ on R^n such that for each compact set K, the restrictions to K of ϕ and of some $\psi \in \mathfrak{D}_\omega$ agree. The topology on $\mathcal{E}_\omega$ is given by the seminorms

$$\phi \to \inf_{\psi = \phi \text{ in } K} \|\psi\|_\lambda \qquad (\forall \lambda > 0; \ \forall K).$$

If $\mathcal{E}'_\omega$ is the strong dual of $\mathcal{E}_\omega$, let $\hat{\mathfrak{D}}_\omega$ and $\hat{\mathcal{E}}'_\omega$ denote the Fourier transforms of the spaces $\mathfrak{D}_\omega$, $\mathcal{E}'_\omega$. For further information on the spaces $\mathfrak{D}_\omega$, $\mathcal{E}_\omega$, etc., see [4], where also the following characterization of $\mathcal{E}_\omega$ can be found (Proposition 1.5.2): $\mathcal{E}_\omega$ is the set of all functions ϕ such that if $\psi \in \mathfrak{D}_\omega$, then $\psi\phi \in \mathfrak{D}_\omega$. The topology in $\mathcal{E}_\omega$ is given by the seminorms $\phi \to \|\psi\phi\|_\lambda$ $(\forall \lambda > 0; \ \forall \psi \in \mathfrak{D}_\omega)$. In other words, $\mathcal{E}_\omega$ may be considered as the space of all multipliers on $\mathfrak{D}_\omega$. For our purposes, however, it is also important to characterize the space $\mathcal{E}'_\omega$ as the space of all convolutors of $\mathfrak{D}_\omega$:

PROPOSITION 1. *The space $\mathcal{E}'_\omega$ consists of all elements Φ in $\mathfrak{D}'_\omega$ whose convolution $\Phi * \phi$ with any element ϕ of $\mathfrak{D}_\omega$ is again in $\mathfrak{D}_\omega$. The topology of $\mathcal{E}'_\omega$ is the compact open topology induced on $\mathcal{E}'_\omega$ from the space $L(\mathfrak{D}_\omega, \mathfrak{D}_\omega)$ of all continuous endomorphisms of $\mathfrak{D}_\omega$.*

Proposition 1 generalizes a result of Ehrenpreis valid for the Schwartz spaces $\mathfrak{D}$, $\mathcal{E}'$ [7, Theorem 5.15]; its proof is also similar.

Let $\mathcal{C}$ be the class of all functions l which are concave, increasing, continuously differentiable on $[0, \infty)$ and such that $l(0) \geqq 0$ and $0 < l'(s) \leqq (2s+1)^{-1}$ for all $s \geqq 0$. Let h be a continuous function on R^1, $-\infty < \inf h(s) \leqq h(s) \nearrow \infty$, and such that its restriction to $[0, \infty)$ is in $\mathcal{C}$. Let $\epsilon(s)$ be a function on R^1, $0 < \epsilon(s) \leqq 1$ for all s, and so rapidly decreasing to zero when $s \to -\infty$, that the inverse function $\rho(s)$ to

$-\log \epsilon(-s)$ is defined and $\rho \in \mathcal{C}$. Finally, let $\mu > 0$. Then the series

$$k(h; \epsilon; \mu; \zeta) = \sum_{s=-\infty}^{\infty} \epsilon(s) e^{|\eta| h(s) - (s+\mu)\omega(\xi)} \qquad (\zeta = \xi + i\eta \in \boldsymbol{C}^n)$$

is locally uniformly convergent in $\boldsymbol{C}^n$. Each function k of this form will be called a majorant. Let $\mathcal{K}_\omega$ be the family of all majorants obtained by varying the parameters h, μ and ϵ. If $\mathcal{A}$ denotes the space of all entire functions in $\boldsymbol{C}^n$ with the topology $\mathfrak{J}$ of uniform convergence on compact sets, let $\mathcal{A}(\mathcal{K}_\omega)$ be the subspace

$$\left\{ g \in \mathcal{A} : \|g\|_k = \sup_{\zeta \in C^n} (|g(\zeta)| / k(\zeta)) < \infty \ (\forall k \in \mathcal{K}_\omega) \right\}$$

equipped with the topology $\mathfrak{J}(\mathcal{K}_\omega)$ given by the norms $\|\cdot\|_k$, $k \in \mathcal{K}_\omega$. Since each majorant k is a positive continuous function in $\boldsymbol{C}^n$, the natural embedding $\mathcal{A}(\mathcal{K}_\omega) \subset \mathcal{A}$ is continuous. Similarly, for each $k \in \mathcal{K}_\omega$, let k^+ be its "Taylor"-part, i.e., $k^+ = \sum_{s=0}^{\infty} \epsilon(s) \exp(\cdots)$. If $\mathcal{K}_\omega^+$ is the family of all k^+, $k \in \mathcal{K}_\omega$, then the subspace $\mathcal{A}(\mathcal{K}_\omega^+)$ of $\mathcal{A}$ and its topology $\mathfrak{J}(\mathcal{K}_\omega^+)$ are defined in an obvious way.

THEOREM 1. *As topological vector spaces $\mathcal{A}(\mathcal{K}_\omega)$ and $\hat{\mathcal{E}}'_\omega$ are identical; similarly $\mathcal{A}(\mathcal{K}_\omega^+) = \hat{\mathfrak{D}}_\omega$.*

For the proof of the second identity, cf. [1], [5]. The proof of the first one uses Proposition 1 and also some estimates of the following type:

LEMMA 1 (CF. [5]). *Let h be a function in $\mathcal{C}$, and p its inverse, $p(0) = 0$. Then, for any $a > 0$, $b \geq 1$,*

$$\max\left(\sum_{s=0}^{\infty} e^{ah(s) - bs}, \sum_{s=0}^{\infty} e^{as - bp(s)} \right) \leqq 4(3 + a) e^{ah(a)}.$$

In the terminology of [7], the first part of Theorem 1 states that the Beurling space $\mathcal{E}_\omega$ is an analytically uniform space (a.u. space) and the family $\mathcal{K}_\omega$ is an a.u. structure for $\mathcal{E}_\omega$. Sometimes it is easier to work with another a.u. structure $\mathcal{K}_\omega^*$ whose elements are integrals

$$k^*(\zeta) = \int_{-\infty}^{\infty} \epsilon(s) \exp[\,|\eta|\, h(s) - s\omega(\xi)]\, ds.$$

In particular, we need the following lower estimate for functions k^*.

LEMMA 2. *Let $\lambda \in \mathcal{C}$, $a > 0$, $b > 0$. If r is the root of the equation $\lambda'(r) = b/a$, then*

$$(1) \qquad \int_0^\infty e^{a\lambda(s)-bs}\,ds \geqq \frac{1}{b}\cdot e^{a\lambda(r)-br},$$

and $r \leqq a/2b$. *If* $\Lambda = \lim_{s\to\infty} \lambda(s) < \infty$, *then for each* δ, $0<\delta<1$, *there are constants* C, t *such that if* $|\eta| \geqq t\omega(\xi)$, *then*

$$(2) \qquad \int_{-\infty}^{\infty} e^{|\eta|\lambda(s)-\omega(\xi)s}\,ds \geqq Ce^{(\Lambda-\delta)|\eta|+\omega(\xi)(1-\delta)};$$

if $\lim_{s\to\infty} \lambda(s) = \infty$, *then any positive number can be taken as* Λ *in* (2).

From here it is easy to derive:

PROPOSITION 2. *Let* $\alpha \in \mathcal{C}$, $\omega(\xi)/\alpha(|\xi|) \to 0$ *for* $|\xi| \to \infty$ *and* $R_\alpha = \{\zeta : |\eta| \geqq \alpha(|\xi|)\}$. *Then outside* R_α *the topologies of* $\hat{\mathfrak{D}}_\omega$ *and* $\hat{\mathcal{E}}'_\omega$ *are the same. More exactly, if* $k \in \mathcal{K}_\omega^+$, *then there is a majorant* $k_1 \in \mathcal{K}_\omega$ *such that* $k = k_1^+$ *and* $k_1(\zeta)/k(\zeta)$ *is bounded on* R_α.

Proposition 2 generalizes to Beurling spaces, a result due to Ehrenpreis and mentioned in §0. It also has analogous consequences; however, these will not be discussed here.

Finally the next assertion complements Theorem 1 and is often useful:

PROPOSITION 3. *For each* $k \in \mathcal{K}_\omega^+$ *with* $\mu = 0$, *let* $W(k)$ *be the set of all* Φ *in* $\mathcal{E}'_\omega$ *such that for some integer* N *and* $C > 0$, $|\hat{\Phi}(\zeta)| \leqq Ce^{N\omega(\xi)}k(\zeta)$ *for all* ζ. *Then the system* $\{W(k)\}_{k\in\mathcal{K}^+}$ *defines a basis of neighborhoods in* $\mathcal{E}'_\omega$.

REMARK. Actually the integer N in Proposition 3 depends only on Φ: either the infimum of all such N is $-\infty$, which is exactly the case of sing supp $\Phi = \varnothing$, i.e., $\Phi \in \mathfrak{D}_\omega$; or in all such inequalities we can take $N = (\text{order of } \Phi) + 1$. This can be seen by Lemma 2.

2. In the rest of the article, only the Schwartz spaces $\mathfrak{D}$, $\mathcal{E}'$ are considered, i.e., $\omega(\xi) = \log(e + |\xi|)$. If $S(R)$ denotes the ball $\{x \in \mathbf{R}^n : |x| < R\}$, then it is immediately clear how to generalize §1 for functions and distributions on $S(R)$. Thus, for each $R > 0$, the corresponding families $\mathcal{K}(R)$ and $\mathcal{K}^+(R)$ can be defined. Given $\Phi \in \mathcal{E}'$, let

$$\mathcal{H}(\Phi) = \{h : \text{for all } \zeta \ |\hat{\Phi}(\zeta)| \leqq Ce^{N\omega(\xi)}k(h;\epsilon;0;\zeta) \quad \text{for some } C, N, \epsilon \text{ and } k \in \mathcal{K}^+(R)\}.$$

If $R(\Phi) = \inf\{R : \operatorname{supp} \Phi \subset S(R)\}$, then

$$(3) \qquad R(\Phi) = \inf_{h\in\mathcal{H}(\Phi)} \lim_{s\to\infty} h(s),$$

as can be easily seen by the Paley-Wiener theorem. Thus formula (3) is not of particular interest. However, it is quite remarkable that by reversing the order of inf and lim in (3), we obtain (after a slight modification) the analogous formula for the singular support. For fixed $\delta > 0$ and $\Phi \in \mathcal{E}'$, let $\mathcal{H}(\Phi; \delta)$ be the class of functions in $\mathcal{H}(\Phi)$ whose derivatives are uniformly small, i.e., for some s_0 depending only on δ, $sh'(s) \leqq \delta$ for all $s \geqq s_0(\delta)$.

THEOREM 2. *For each* $\Phi \in \mathcal{E}'$, *let* $r(\Phi) = \inf\{R: \text{sing supp } \Phi \subset S(R)\}$. *Then*

$$r(\Phi) = \lim_{\delta \to 0} \lim_{s \to \infty} \inf_{h \in \mathcal{H}(\Phi;\, \delta)} h(s + N). \tag{4}$$

The proof is based on the use of the inverse Fourier transform and, for this reason, does not seem to apply to Beurling classes. One can also derive a similar formula directly for the supporting function of the set sing supp Φ. As it is easy to see, the function inf $h(s+N)$ occurring in formula (4) is increasing in both variables s and δ. Therefore both limits in (4) can be replaced by suprema and their order reversed. The proof of Theorem 2 and some of its applications will be given in [3].

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