BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

BOOK REVIEWS

Equations in Linear Spaces, by D. Przeworska-Rolewicz and S. Rolewicz. Polish Scientific Publishers, Warsaw, 1968, 380 pp. $15.00; *Theory of Generalized Spectral Operators*, by Ion Colojoara and Ciprian Foiaş. Gordon and Breach, New York, 1968, xvi+232 pp. $14.50; *Analyse harmonique des operateurs de l'espace de Hilbert*, by Bela Sz.-Nagy and Ciprian Foiaş. Masson, Paris, 1967, xi+373 pp.; *Introduction to the Theory of Linear Nonselfadjoint Operators*, by I. C. Goh'berg and M. G. Kreĭn. American Mathematical Society, Providence, 1969, xv+378 pp.

The theory of (mostly) bounded linear operators is the subject of the four books under review. None is, strictly speaking, a textbook; each is rather a monograph on an area of operator theory to which that pair of authors has contributed. The intersection of the books is small and the points of view quite different. All, but the second, present comparatively well-developed theories of recent origin and attempt to reach a wider audience than the specialist. The second book presents a tentative systematization of the rapidly growing study of generalized spectral theory. The last book is the first of three volumes of a very ambitious undertaking with the goal of presenting a large part of the work of the last two decades in operator theory to which the Russians have made crucial contributions.

The earliest results in operator theory were perhaps those attained by Volterra and Fredholm in the study of integral equations. After the extensive contributions of Hilbert, an abstract formulation of certain of the results concerning compact operators was given by F. Riesz in the so called "Fredholm alternative." Later, F. Noether showed using regularization that certain singular integral equations gave rise to Fredholm operators and proved the first "index theorem." The characterization of these operators as the operators which are invertible modulo the compact operators was given by Atkinson. Finally, a detailed and far reaching systematization of the study of this class of operators was made by Goh'berg and Kreĭn. It is mainly an exposition and refinement of this line of development with which the Rolewiczs' book is concerned.

As the title indicates the authors study the problem of solving the equation $Af=g$ for A a linear operator. Linear operators in linear spaces and especially nearly invertible operators or sets of operators are studied in the first third of the book. Algebraic operators are similarly studied. A nearly self-contained treatment of continuous linear operators on linear topological spaces is given in the second part with the definition of the classes of compact and Fredholm operators and the development of the Riesz theory for compact operators. Operators on Banach spaces are studied in the third part along with the theory of Banach spaces. After introducing various classes of compact operators, the authors present the Fredholm theory including the analytical index and its properties. The book concludes with various applications to integral equations including systems.

In summary the authors present a detailed, nearly self-contained account of the theory of "nearly invertible," compact, and Fredholm operators on linear topological spaces along with a few applications. The reader interested in knowing just what portions of this theory generalize to his favorite class of spaces will find this book extremely useful. The reader trying to learn the basic facts of this theory will need considerable patience, however, and might be advised to first look elsewhere. A book which presented the fundamentals more succinctly with the extensions to more general spaces reserved for the fine print and with more examples and applications would have been preferred by this reviewer.

The second book continues the line of development which began with the spectral theorem. Following the formulation by Hilbert of the spectral theorem for bounded selfadjoint operators, various proofs and significant generalizations of this result were given by Riesz, von Neumann, Stone and others. Somewhat later a different point of view toward this theory was adopted by Yoshida and then Dunford. Due, in part, to the mode of proof, the functional calculus came to play a fundamental role. Operators on Banach spaces were studied for which the existence of a functional calculus based on the Banach algebra of continuous functions on the spectrum was postulated. The successes of the Dunford spectral theory were mitigated, in part, by the difficulty in showing that concrete operators were spectral. Finally, the fact that several natural classes of operators were only "almost spectral" led to the consideration of various generalizations. Previous to this, functional calculi based on other algebras of functions had appeared implicitly in the study of various classes of operators and occasionally explicitly. Recently, many authors have attempted the systematic study of various classes of operators assumed to possess a functional calculus based on certain algebras of functions.

In the second book the authors attempt to give a comprehensive account of this rapidly developing field with particular emphasis on their own work. A systematic presentation of the Dunford theory is neither attempted nor assumed. Rather a very general framework is given for the study of perhaps the broadest class of operators retaining the flavor of spectral theory. An operator T on a Banach space $\mathfrak{X}$ is said to be decomposable if there exists a finitely additive subspace valued spectral decomposition $\mathfrak{X}_K$ defined on the compact subsets K of the complex plane such that $\sigma(T|\mathfrak{X}_K)\subset K$. (In the Dunford theory the subspace $\mathfrak{X}_K$ is replaced by a projection E_K onto it and E_F is assumed to be defined for Borel sets and to be countably additive.) Such a spectral decomposition is uniquely determined by T but the converse problem is more difficult. Operators T_1 and T_2 are said to be quasi-nilpotent equivalent, denoted $T_1 \overset{q}{\sim} T_2$, if

$$\left\|\sum_{k=0}^{n}(-1)^n\binom{n}{k}T_1^kT_2^{n-k}\right\|^{1/n}+\left\|\sum_{k=0}^{n}(-1)^n\binom{n}{k}T_2^kT_1^{n-k}\right\|^{1/n}\to 0.$$

If T_1 and T_2 commute, then $T_1 \overset{q}{\sim} T_2$ if and only if T_1-T_2 is quasi-nilpotent but T_1-T_2 does not have to be quasi-nilpotent for $T_1 \overset{q}{\sim} T_2$. The authors show that if $T_1 \overset{q}{\sim} T_2$, then T_1 is decomposable if and only if T_2 is decomposable and their spectral decompositions coincide. Conversely, if T_1 and T_2 are decomposable with the same spectral decomposition, then $T_1 \overset{q}{\sim} T_2$.

Generalized spectral operators are operators for which there exists a continuous C^∞-functional calculus. Such operators are decomposable and the spectral distribution is of finite order in an appropriate sense. In particular, a Dunford spectral operator is generalized spectral if and only if the quasi-nilpotent part is actually nilpotent. A refined notion of generalized spectral operator is introduced for which the spectral distribution is almost unique and generalized spectral operators with "thin spectrum" are shown to belong to this class. With the aid of the tensor product, it is shown that the sum and product of commuting generalized spectral operators are also generalized spectral operators.

In the longest chapter the authors consider generalized spectral operators with spectrum contained in either the unit circle or the real line. Various relations due to F. Wolf between the operators with unitary spectrum and growth conditions on the resolvent and powers of the operator are obtained. Results analogous to those of Wermer concerning operators related to certain Banach algebras introduced by Beurling are given. Further, results of Kantorowicz, Smart and

Tillman which concern the growth of the resolvent of generalized spectral operators with real spectrum are synthesized and extended. Operators with real spectrum and imaginary part lying in a p-ideal of the compact operators as well as the similar class for the circle are shown to fit into this context. Lastly, various theorems on triangular forms for operators and existence theorems for invariant subspaces are studied.

In summary the authors have synthesized and extended a diverse number of results relating to a common theme with proofs presented in complete detail. As we mentioned earlier this is a rapidly developing area and the book represents more of an interim report than a polished product. The book is a good source for recent ideas and gives some idea of what needs to be done. The book's most serious shortcoming is the lack of consideration of operators obtained from differential and integral equations. (The authors have informed the reviewer that the material from Remark V 5.4 to the end of §5 is incorrect.)

The third book gives an account of a theory in which several lines of development have joined: the seminal work of Beurling indicating the importance of the Hardy spaces and analytic functions in the study of the shift operator and the generalizations due to Lax and Halmos; the study of nearly selfadjoint operators by Livsič and Brodskii using the characteristic operator function including the work of Potapov; the study of Helson and Lowdenslager of prediction theory in several variables and analytic functions; the canonical models based on spaces of formal power series of deBrange and Rovnyak; and the study of Sz.-Nagy and Foiaş beginning with the result of the first author on the existence and uniqueness of unitary dilations and continuing through the detailed explication of the relation between a contraction and its unitary dilation.

All of these developments are different facets of one theory. The point of view adopted in this book is quite naturally that of making the unitary dilation paramount, but an account is given of most of the other points of view. The book is complete, systematic, and self-contained and the bibliography and notes provide a good orientation to the literature.

A very subtle functional calculus plays an important role in these studies which extends the functional calculus due to Riesz. In contrast to the latter, however, this functional calculus does not require analyticity on a neighborhood of the spectrum but essentially only on the interior. One of the major successes of the theory is the complete generalization of the Jordan theory for matrices to operators which are the zero of some analytic function. (Unfortunately, many of the latter results have appeared after the book but the basic theory and references are still in the book.)

A second major success is the study of contractions which are nearly unitary. Using their model theory the authors have been able to simplify and to extend the results on when such operators are similar to unitary operators or equivalently, when an accretive or dissipative operator is similar to a selfadjoint operator. This is a study to which Sahnovič and other Russian authors have made important contributions.

In summary the authors have given a readable and useful account of an important theory with roots in several areas to which they have been principal contributors. A second edition in English has appeared which incorporates some of the more recent work including the lifting theorem and its applications. The book will surely be a standard work in operator theory and related areas for the years to come.

As we mentioned previously, the last book is the first of a projected series of three which will eventually cover a large portion of the theory of nonselfadjoint operators. As such there will be considerable overlap with the first three books reviewed. In this first volume, there is not, and the authors main concern is the systematic study of various selfadjoint ideals of compact operators and of the problem of the completeness of the root subspaces for various classes of nearly selfadjoint operators.

The ideal of Hilbert-Schmidt operators appeared in the earliest studies as the integral operators with square integrable kernels. The other p-ideals for $1 \leqq p < \infty$ were studied by von Neumann and Schatten and have played a fundamental role in many investigations. Various ideals analogous to Orlicz spaces have appeared in studies such as those of Macaev on nonselfadjoint operators. In the first one-third of the book the authors offer a systematic study of such selfadjoint ideals of compact operators. Characteristically the authors have both deepened and refined the subject and these chapters will undoubtedly become the standard reference.

The notion of s-number is made basic and a systematic treatment of various inequalities due to Weyl, K. Fan, and A. Horn relating s-numbers and eigenvalues are presented. The trace class of nuclear operators receive special treatment. Various tests for the nuclearity of an operator are presented along with the fundamental result of Lidskii that the trace is equal to the sum of the eigenvalues. Perturbation determinants are also studied along with various resolvent estimates stemming from Carleman for operators having an imaginary part lying in a p-ideal. The latter results make extensive use of the theory of analytic functions and especially of entire functions.

The principal application of these results in this volume is to the completeness problem of root spaces. Approximately one-third of the book is devoted to this problem along with a study of the various kinds of bases which can exist in Hilbert space and the notion of expansion appropriate to each. The early and fundamental results of M. V. Keldys in this area are presented in complete detail. A rather thorough presentation of this area is given with various techniques being illustrated. Lastly, various asymptotic properties of the spectrum of weakly perturbed operators are given.

In summary this book is a thorough and complete treatment along with many worthy contributions of some important but relatively neglected areas of abstract operator theory with applications to more immediate and concrete problems. We eagerly await the remaining two volumes.

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Introduction to analytic number theory, by K. Chandrasekharan, Springer, 1968; *Arithmetic functions*, by K. Chandrasekharan, Springer, 1970; *Multiplicative number theory*, by Harold Davenport, Markham, 1967; *Sequences*, by H. Halberstam and K. F. Roth, Oxford University Press, 1966.

Recent years have seen an explosion in the number of books in most branches of mathematics and this is true of number theory. Most books contain little that is new, even in book form. This is the case of 8/3 of the four books in this review. They are well written and make good textbooks and pleasant reading but they are not revolutionary. The remaining 4/3 books are new in book form and we will spend most of the review on these.

We begin this review with a discussion of Chandrasekharan's *Introduction to analytic number theory*, which is a translation with some slight revisions of the author's *Einführung in die analytische Zahlentheorie* (Springer lecture notes series number 29). This book presupposes the usual knowledge of functions of a complex variable (i.e. Cauchy's theorem) but virtually no knowledge of number theory. Indeed, the book begins with the unique factorization theorem and in the early chapters moves through (among other things) congruences, the law of quadratic reciprocity and several standard arithmetical functions. The later chapters include Weyl's theorems on uniform distribution, Minkowski's convex body theorem, Dirichlet's theorem