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ASYMPTOTICS OF A NONLINEAR RELATIVISTIC WAVE EQUATION[1]

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K. Jörgens [1] has proved the global existence of classical solutions of the Cauchy problem for

$$u_{tt} - \Delta u + m^2 u + u^3 = 0 \tag{*}$$

with $m \geqq 0$, in all space-time. I. Segal in [3] has proved the existence of the free-to-perturbed wave operators and in [4] the existence of the scattering operator on numerically small solutions. He has conjectured that the scattering operator exists in general. Segal's conjecture has been verified when $m=0$ in [5]. We have succeeded in proving the conjecture when $m>0$.

DEFINITIONS. By a *free solution* we mean a solution of the associated linear equation (equation (*) without u^3). The norm

$$\|u\|^2 = \sup_t \int (u_t^2 + |\nabla u|^2 + m^2 u^2)\, dx + \sup_{x,t} u^2 + \int_{\infty}^{\infty} \sup_x u^2\, dt$$

is finite for all free solutions with smooth Cauchy data of compact support. Define F to be the space of all their limits under this norm.

THEOREM 1. *Any solution of (*) with smooth Cauchy data of compact support tends to zero uniformly as $|t| \to \infty$. Furthermore, the solution approaches a free solution u_+ in the energy norm as $t \to +\infty$ and a free solution u_- as $t \to -\infty$.*

There are extensions of this theorem to: weak solutions, more general nonlinear terms, and a rate of decay as $|t| \to \infty$.

THEOREM 2. *Whenever u is a solution of (*) which tends to $u_\pm$ as above, we define the operator S by $S(u_-) = u_+$. Then S is defined on all of F and is a homeomorphism of this space onto itself which preserves the energy norm.*

The proofs are based on an estimate derived from [2], on some new

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estimates of the Riemann function of the linear equation, and on Segal's approach to the scattering problem.

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