

COMPLETENESS OF THE WAVE OPERATORS FOR SCATTERING PROBLEMS OF CLASSICAL PHYSICS¹

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ABSTRACT. Many wave propagation phenomena of classical physics are governed by equations of the Schrödinger form $iD_t u = \Lambda u$ where

$$(1) \quad \Lambda = -iE(x)^{-1} \sum_{j=1}^n A_j D_j.$$

($E(x)$ and the A_j are Hermitian matrices, $E(x)$ is positive definite and the A_j are constant.) The time-evolution of such phenomena is described by a group of unitary operators $\exp(-it\Lambda)$ on the Hilbert space \mathcal{H} with the energy norm

$$(2) \quad \|u\|_E^2 = \int_{R^n} u(x)^* E(x) u(x) dx.$$

If $E(x)$ is replaced by a constant E_0 the corresponding space and operator are denoted by H_0 and Λ_0 . In this paper it is shown that the wave operators

$$(3) \quad W_{\pm}(\Lambda, \Lambda_0, J) = \text{s-lim}_{t \rightarrow \pm \infty} \exp(it\Lambda) J \exp(-it\Lambda_0) P_0^{\text{ac}}$$

exist and are complete if $\Lambda_0(p) = E_0^{-1} \sum_{j=1}^n A_j p_j$ satisfies

$$(4) \quad \text{rank } \Lambda_0(p) = m - k \quad \text{for all } p \in R^n - \{0\},$$

$E(x)$ and $D_j E(x)$ are continuous and bounded ($j=1, 2, \dots, n$), $E(x)$ is uniformly positive definite, $\lim_{|x| \rightarrow \infty} E(x) = E_0$ uniformly in $x/|x|$ and

$$(5) \quad \int_{R^n} (1 + |x|^2)^{\mu} |E(x) - E_0|^2 dx < \infty \quad \text{for some } \mu > n/2.$$

(In (3), $J: H_0 \rightarrow H$ is the identification map; $Ju = u$, and P_0^{ac} is the orthogonal projection onto the absolutely continuous subspace for Λ_0 . W_{\pm} are complete if their ranges equal H^{ac} , the absolutely continuous subspace for Λ .)

1. Wave operators. An abstract theory of scattering with two

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Hilbert spaces was formulated by Wilcox in [11] as an operator-theoretic description of the scattering phenomena of classical physics, and was subsequently developed by T. Kato [7], A. L. Belopol'skiĭ and M. Š. Birman [2] and the authors [9]. In this theory the instantaneous states of a wave in a medium are described by the vectors of a Hilbert space \mathcal{H} and the time-evolution of the wave is described by a group of unitary operators $e^{-it\Lambda}$ on \mathcal{H} with selfadjoint generator Λ . The scattering states of the medium are described by the vectors of $\mathcal{H}^{\text{ac}} = P^{\text{ac}}\mathcal{H}$, the subspace of absolute continuity for Λ [6, p. 516] with corresponding orthogonal projection P^{ac} .

Scattering theory is concerned with the asymptotic behavior for $t \rightarrow \pm \infty$ of the waves in a medium. This may be studied by comparing these waves with those in a second (simpler) medium with state space \mathcal{H}_0 and generator Λ_0 . Let $J: \mathcal{H}_0 \rightarrow \mathcal{H}$ be a bounded linear operator which maps \mathcal{H}_0 onto \mathcal{H} and define the wave operators for the triple (Λ, Λ_0, J) by

$$(1.1) \quad W_{\pm}(\Lambda, \Lambda_0, J) = \text{s-lim}_{t \rightarrow \pm \infty} e^{it\Lambda} J e^{-it\Lambda_0} P_0^{\text{ac}},$$

when these limits exist (s-lim denotes strong limit). Each scattered wave for Λ_0 is asymptotically equal, for $t \rightarrow \pm \infty$, to a corresponding wave for Λ if and only if these wave operators exist [11]. Moreover, the wave operators (1.1) map \mathcal{H}_0 into \mathcal{H} and are partial isometries [6, p. 258] with initial set $\mathcal{H}_0^{\text{ac}}$ and final set contained in \mathcal{H}^{ac} [2], [7], [11]. The wave operators are said to be complete if the final set is \mathcal{H}^{ac} . In this case it is known [2], [7], [11] that the wave operators $W_{\pm}(\Lambda_0, \Lambda, J^*)$ also exist and are complete, and

$$(1.2) \quad W_{\pm}(\Lambda, \Lambda_0, J)^* = W(\Lambda_0, \Lambda, J^*).$$

The principal goal of the abstract theory of scattering is to find criteria for the existence and completeness of the wave operators. The following theorem was proved by the authors in [9].

THEOREM 1.1. *Let Λ and Λ_0 satisfy*

$$(1.3) \quad JD(\Lambda_0) = D(\Lambda),$$

$$(1.4) \quad J^*P^{\text{ac}}D(\Lambda) \subset P_0^{\text{ac}}D(\Lambda_0),$$

$$(1.5) \quad \Pi(\delta)(\Lambda J - J\Lambda_0)\Pi_0(\delta) \in B_1(\mathcal{H}_0, \mathcal{H}), \text{ and}$$

$$(1.6) \quad (J^*J - 1)P_0^{\text{ac}}\Pi_0(\delta) \in B_0(\mathcal{H}_0),$$

for each bounded interval $\delta \subset \mathbb{R}^1$. Then the wave operators (1.1) exist and are complete.

In Theorem 1.1, $D(\Lambda)$ and $D(\Lambda_0)$ denote the domains of Λ and Λ_0 , and $\Pi(\delta)$ and $\Pi_0(\delta)$ their spectral measures [6, p. 355]. $B_1(\mathcal{H}_0, \mathcal{H})$

denotes the trace class of operators from \mathfrak{K}_0 to \mathfrak{K} and $B_0(\mathfrak{K}_0)$ the class of compact operators on \mathfrak{K}_0 [6, p. 519]. The theorem is based on the theory of local wave operators developed by M. Š. Birman [2], [3].

Theorem 1.1 was used in [9] to prove the completeness of the wave operators for a class of scattering problems of classical physics. The purpose of this announcement is to show that Theorem 1.1 can be applied to a significantly larger class of such problems.

2. Wave equations of classical physics. Many wave propagation phenomena of classical physics are governed by equations of the Schrödinger form $iD_t u = \Lambda u$ where Λ is a matrix partial differential operator:

$$(2.1) \quad \Lambda = -iE(x)^{-1} \sum_{j=1}^n A_j D_j,$$

$x = (x_1, x_2, \dots, x_n) \in R^n$, $t \in R^1$, $D_t = \partial/\partial t$, $D_j = \partial/\partial x_j$, $u = u(x, t)$ is an $m \times 1$ (column) matrix, $E(x)$ and A_1, A_2, \dots, A_n are $m \times m$ Hermitian matrices and $E(x)$ is positive definite. Examples include electromagnetic waves, acoustic waves, elastic waves and other less familiar phenomena [11].

The matrix-valued function $E(x)$ is assumed below to have the following additional properties.

(2.2) $E(x)$ is uniformly positive definite for $x \in R^n$; that is, the smallest eigenvalue of $E(x)$ is bounded below by a positive constant.

(2.3) $E(x), D_1 E(x), D_2 E(x), \dots, D_n E(x)$ are continuous and bounded for $x \in R^n$.

It follows that (if M^* denotes the conjugate transpose)

$$(2.4) \quad (u, v)_E = \int_{R^n} u(x)^* E(x) v(x) dx$$

determines an inner product and Hilbert space $\mathfrak{K} = \{u: u(x) \text{ is Lebesgue measurable on } R^n \text{ and } (u, u)_E < \infty\}$. Moreover, the restriction of Λ to $C_0^1(R^n; C^m) \subset \mathfrak{K}$ is a symmetric operator on \mathfrak{K} which is essentially selfadjoint [8]. The unique selfadjoint extension of this operator is also denoted by Λ below.

3. The completeness theorem. Assume that

$$(3.1) \quad \lim_{|x| \rightarrow \infty} E(x) = E_0 \quad \text{uniformly in } x/|x|.$$

Then E_0 is positive definite by (2.2). Let \mathcal{H}_0 and Λ_0 be the Hilbert space and operator (2.1) determined by E_0 . Then \mathcal{H} and \mathcal{H}_0 are equivalent [9] and $J: \mathcal{H}_0 \rightarrow \mathcal{H}$, $Ju = u$ for all $u \in \mathcal{H}_0$, is a bounded operator which carries \mathcal{H}_0 onto \mathcal{H} .

The existence and completeness of $W_{\pm}(\Lambda, \Lambda_0, J)$ were proved in [9] under assumptions (2.2), (2.3), (3.1),

$$(3.2) \quad \int_{R^n} (1 + |x|^2)^{\mu} |E(x) - E_0|^2 dx < \infty \quad \text{for some } \mu > n/2,$$

and

$$(3.3) \quad \Lambda_0 \text{ is a uniformly propagative operator.}$$

Condition (3.3) means that the distinct eigenvalues of

$$(3.4) \quad \Lambda_0(p) = E_0^{-1} \sum_{j=1}^n A_j p_j$$

have constant multiplicity and constant algebraic sign for all $p \in R^n - \{0\}$. The purpose of this announcement is to strengthen the completeness theorem of [9] by replacing (3.3) by the weaker condition

$$(3.5) \quad \text{rank } \Lambda_0(p) = m - k \quad \text{for all } p \in R^n - \{0\}.$$

Operators Λ_0 with this property are said to have constant deficit k [8]. The new theorem may be stated as follows.

THEOREM 3.1. *The wave operators $W_{\pm}(\Lambda, \Lambda_0, J)$ exist and are complete if Λ , Λ_0 and J are the operators defined above, $E(x)$ and E_0 satisfy (2.2), (2.3), (3.1) and (3.2) and Λ_0 has constant deficit.*

Condition (3.5) holds if and only if the nonzero eigenvalues of (3.4) are bounded away from zero when $|p| = 1$ [10]. Thus operators Λ_0 with constant deficit k are precisely those for which the zero eigenvalue of $\Lambda_0(p)$ has constant multiplicity k for all $p \in R^n - \{0\}$. Important examples of operators with constant deficit which are not uniformly propagative occur in the theory of electromagnetic and elastic wave propagation in crystals [4, p. 602], [5]. An example which does not have constant deficit is provided by Alfvén waves in magnetohydrodynamics [4, pp. 612–617].

4. Proof of the completeness theorem. Theorem 3.1 was proved in [9], under the additional hypothesis (3.3), by using a coerciveness theorem for Λ given in [8] and a construction of the spectral family $\Pi_0(\lambda)$ for Λ_0 given in [12] to verify hypotheses (1.3)–(1.6) of Theorem 1.1. These results were based on (3.3). The proof given in [9]

also suffices to prove Theorem 3.1 if the coerciveness theorem of [10], based on (3.5), is used together with the following results.

THEOREM 4.1. *Let $\lambda_1(p) \geq \lambda_2(p) \geq \dots \geq \lambda_m(p)$ be the repeated eigenvalues of $\Lambda_0(p)$, ordered as indicated for all $p \in R^n$. Then each $\lambda_j(p)$ is continuous on R^n . Moreover, there exist m functions $r_j: R^n \rightarrow C^m$ which are Lebesgue measurable on R^n and satisfy $\Lambda_0(p)r_j(p) = \lambda_j(p)r_j(p)$ and $r_j(p)^* E_0 r_k(p) = \delta_{jk}$ for $1 \leq j, k \leq m$ and almost all $x \in R^n$.*

THEOREM 4.2. *The spectral family for Λ_0 can be written $\Pi_0(\lambda) = \Phi^* \hat{\Pi}_0(\lambda) \Phi$ where Φ is the Fourier transform on \mathcal{H}_0 with inverse Φ^* and (if $H(\lambda)$ is the Heaviside function)*

$$(4.1) \quad \hat{\Pi}_0(\lambda) = \sum_{j=1}^m H(\lambda - \lambda_j(\cdot)) r_j(\cdot) r_j(\cdot)^* E_0.$$

Theorem 4.1 was proved by Wilcox in [13]. Theorem 4.2 was proved by Wilcox in [12] under the additional hypothesis (3.3). It was proved by G. S. S. Avila [1] without this hypothesis on the assumption that Theorem 4.1 was true. Hypothesis (3.5) is not needed for Theorems 4.1 and 4.2.

Hypothesis (3.5) is used in [10] in the proof of the coerciveness theorem and again, in the proof of (1.5) and (1.6), to show that the nonzero eigenvalues of $\Lambda_0(p)$ are bounded away from zero (see the proof of Lemma 5.2 in [9]).

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