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BERGMAN OPERATORS FOR ELLIPTIC EQUATIONS IN THREE INDEPENDENT VARIABLES

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Introduction. S. Bergman [1] and I. N. Vekua [7] have both constructed integral operators which map analytic functions of one complex variable onto solutions of the elliptic equation

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1}$$

We wish to announce in this note the extension of these results to the three-variable case, i.e. the equation

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} + a(x, y, z)u_x + b(x, y, z)u_y \\ + c(x, y, z)u_z + d(x, y, z)u = 0 \end{aligned} \tag{2}$$

where a, b, c, d are real valued entire functions of the (complex) variables x, y, z. (With minor modifications we could have assumed only that a, b, c, d are analytic in some ball containing the origin.) Partial results on integral operators for equation (2) (in the special case when $a=b=c=0$) have been obtained by Bergman [1], Tjong [6], Colton and Gilbert [4], and Gilbert and Lo [5].

Main results. Let $X=x$, $Z=\frac{1}{2}(y+iz)$, $Z^*=\frac{1}{2}(-y+iz)$. Then equation (2) becomes

$$\begin{aligned} U_{XX} - U_{ZZ^*} + A(X, Z, Z^*)U_X + B(X, Z,Z^*)U_Z \\ + C(X, Z, Z^*)U_{Z^*} + D(X, Z, Z^*)U = 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} U(X, Z, Z^*) &= u(x, y, z), \\ A(X, Z, Z^*) &= a(x, y, z), \\ B(X, Z, Z^*) &= \tfrac{1}{2}(b(x, y, z) + ic(x, y, z)) \\ C(X, Z, Z^*) &= \tfrac{1}{2}(-b(x, y, z) + ic(x, y, z)) \\ D(X, Z, Z^*) &= d(x, y, z). \end{aligned} \tag{4}$$

The substitution

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$$V(X, Z, Z^*) = U(X, Z, Z^*) \exp\left[- \int_0^Z C(X, Z', Z^*)\, dZ' \right] \tag{5}$$

yields the following equation for $V(X, Z, Z^*)$,

$$\begin{aligned} V_{XX} - V_{ZZ^*} + \tilde{A}(X, Z, Z^*)V_X + \tilde{B}(X, Z, Z^*)V_Z \\ + \tilde{D}(X, Z, Z^*)V = 0, \end{aligned} \tag{6}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ are expressible in terms of the coefficients A, B, C, D. Let $U_0(X, Z, Z^*)$ be the real valued, entire solution of equation (3) which satisfies the Goursat data $U_0(X, 0, Z^*) = U_0(X, Z, 0) = 1$. Note that in the special case when $D = 0$ we can choose $U_0 \equiv 1$. In the general case when $D \neq 0$, U_0 can be constructed via the recursive scheme

$$U_0 = 1 + \lim_{n\to\infty} W_n,$$

$$\begin{aligned} W_{n+1} = \int_0^Z \int_0^{Z^*} \bigg(\frac{\partial^2 W_n}{\partial X^2} + A \frac{\partial W_n}{\partial X} + B \frac{\partial W_n}{\partial Z} \\ + C \frac{\partial W_n}{\partial Z^*} + DW_n - D \bigg)\, dZ' dZ^{*\prime}, \end{aligned} \tag{7}$$

$$W_0 = 0.$$

By introducing the variables

$$\begin{aligned} \xi_1 &= 2\zeta Z, \\ \xi_2 &= X + 2\zeta Z, \\ \xi_3 &= X + 2\zeta^{-1} Z^*, \\ \mu &= \tfrac{1}{2}(\xi_2 + \xi_3) = X + \zeta Z + \zeta^{-1} Z^*, \end{aligned} \tag{8}$$

where ζ is a complex variable such that $1-\epsilon < |\zeta| < 1+\epsilon$, $0 < \epsilon < \frac{1}{2}$, we can now state the following theorem. In the theorems which follow "Re" denotes "take the real part" and "Im" denotes "take the imaginary part."

THEOREM 1. *Let*

$$E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) \tag{9}$$

where

$$p_1^{(n+1)} - \tfrac{1}{2}(\tilde{A}^* + \tilde{B}^*\zeta)p^{(n+1)}$$

$$= \frac{1}{2n+1}\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} + (\tilde{A}^* + 2\tilde{B}^*\zeta)p_2^{(n)} + \tilde{A}^*p_3^{(n)} + 2\tilde{B}^*\zeta p_1^{(n)} + \tilde{D}^*p^{(n)}\},$$

(10)

$$p^{(1)}(\xi_1, \xi_2, \xi_3, \zeta) = \exp\left[\frac{1}{2}\int_0^{\xi_1}(\tilde{A}^* + \tilde{B}^*\zeta)\, d\xi_1'\right],$$

$$p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0, \quad n = 1, 2, \cdots,$$

$$p_i^{(n)} = \partial p^{(n)}/\partial\xi_i, \qquad p_{ij}^{(n)} = \partial^2 p^{(n)}/\partial\xi_i\partial\xi_j,$$

with $\tilde{A}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{A}(X, Z, Z^*)$, $\tilde{B}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{B}(X, Z, Z^*)$, $\tilde{D}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{D}(X, Z, Z^*)$. *Then the following is true*:

(1) $E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = E(X, Z, Z^*, \zeta, t)$ *is regular in* $G_R \times B \times T$ *where* $G_R = \{(\xi_1, \xi_2, \xi_3) : |\xi_i| < R, i = 1, 2, 3\}$, $B = \{\zeta : 1-\epsilon < |\zeta| < 1+\epsilon\}$, $T = \{t : |t| \leq 1\}$, *and R is an arbitrarily large positive number.*

(2) *If* $U(X, Z, Z^*)$ *is a real valued (for* (x, y, z) *real) solution of equation* (3) *which is regular in some neighborhood of the origin, then there exists an analytic function* $f(\mu, \zeta)$ *which is regular for* μ *in some neighborhood of the origin and* $|\zeta| < 1+\epsilon$, *such that locally*

(11) $$U(X, Z, Z^*) = U(0, 0, 0)U_0(X, Z, Z^*) + \mathrm{Re}\ \mathbf{C}_3\{f\},$$

where

(12) $$\mathbf{C}_3\{f\} = \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{+1}\exp\left[\int_0^Z C(X, Z', Z^*)\, dZ'\right] \cdot E(X, Z, Z^*, \zeta, t) f(\mu(1-t^2), \zeta)\frac{dt}{(1-t^2)^{1/2}}\frac{d\zeta}{\zeta}.$$

(3) *If*

(13) $$U(X, 0, Z^*) - U(0, 0, 0) = \sum_{n=0}^{\infty}\sum_{m=0;\, n+m\neq 0}^{\infty}\gamma_{nm}X^nZ^{*m},$$

(14) $$\overline{C(X, Z, Z^*)} = \overline{C(X, -Z^*, -Z)}, \qquad x, y, z \text{ real},$$

then

(15) $$f(\mu, \zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} a_{nm}\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})\Gamma(\frac{3}{2})}\mu^n\zeta^m,$$

where

$$a_{n-1,0} = \gamma_{n0}, \qquad n \geqq 1,$$

$$(16) \quad \begin{aligned} a_{n+m-1,m} &= \frac{2n!m!}{(n+m)!}\gamma_{nm} - \sum_{k=0}^{n-1} \frac{n!}{(n+m)!k!}\delta_{km}\gamma_{n-k,0}, \quad n \geqq 0, m > 0, \\ \delta_{km} &= \left(\frac{\partial^{k+m}}{\partial X^k \partial Z^{*m}} \exp\left[\int_0^{-Z^*} \tilde{C}(X, Z', 0)\, dZ'\right]\right)_{X=Z^*=0}. \end{aligned}$$

(*The finite series in equation* (16) *is omitted when* $n = 0$.)

The fact that *every* real valued twice continuously differentiable solution of equation (2) (i.e., a regular solution of equation (3)) can be represented in the form of equation (11) now leads to the following theorem:

THEOREM 2. *Let* G *be a bounded, simply connected domain in Euclidean three space, and, for* x, y, z *real, define*

$$\begin{aligned} u_0(x, y, z) &= U_0(X, Z, Z^*) \\ (17) \quad u_{2n,m}(x, y, z) &= \operatorname{Re} \mathbf{C}_3\{\mu^n \zeta^m\}, \quad 0 \leqq n < \infty, m = 0, 1, \cdots, n+1, \\ u_{2n+1,m}(x, y, z) &= \operatorname{Im} \mathbf{C}_3\{\mu^n \zeta^m\}, \quad 0 \leqq n < \infty, m = 0, 1, \cdots, n+1. \end{aligned}$$

Then the set $\{u_0\} \cup \{u_{nm}\}$ *is a complete family of solutions for equation* (2) *in the space of real valued solutions of equation* (2) *defined in* G.

Special cases. (a) $A = B = C = 0$.

THEOREM 3. *Assume* $A = B = C = 0$, *and let*

$$(18) \quad \tilde{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n p^{(n+1)}(\xi_1, \xi_2, \xi_3, \zeta)$$

where the $p^{(n)}$ *are defined by equation* (10) *with* $\tilde{A} = \tilde{B} = 0$. *Then*

(1) $\tilde{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \tilde{E}(X, Z, Z^*, \zeta, t)$ *is regular in* $G_R \times B \times T$.

(2) *Every real valued solution* $U(X, Z, Z^*)$ *of equation* (3) *which is regular in some neighborhood of the origin can be represented locally in the form*

$$(19) \quad U(X, Z, Z^*) = \operatorname{Re} \mathbf{P}_3\{f\}$$

where

$$(20) \quad \mathbf{P}_3\{f\} = \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{+1} \tilde{E}(X, Z, Z^*, \zeta, t) f(\mu(1-t^2), \zeta) \frac{dt}{(1-t^2)^{1/2}} \frac{d\zeta}{\zeta},$$

and

$$f(\mu, \zeta) = -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1-t^2), \zeta)\frac{dt}{t^2} \tag{21}$$

$$g(\mu, \zeta) = 2\frac{\partial}{\partial\mu}\left[\mu\int_0^1 U(t\mu, 0, (1-t)\mu\zeta)\,dt\right] - U(\mu, 0, 0). \tag{22}$$

In equation (21) γ' *is a rectifiable arc joining the points* $t=-1$ *and* $t=+1$ *and not passing through the origin.*

(b) $A=B=C=D=0$.

In the special case when $A=B=C=D=0$, the operator $\boldsymbol{P}_3$ reduces to the well-known Bergman-Whittaker operator $\boldsymbol{B}_3$ [1] and equation (22) gives a new inversion formula for the operator Re $\boldsymbol{B}_3$.

Complete proofs of the results stated in this announcement will appear in [2] and [3].

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