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ON THE NONEXISTENCE OF COMPLEX HAAR SYSTEMS[1]

BY JOHN M. OVERDECK

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1. **Introduction.** Schoenberg and Yang [8] have shown that a finite polyhedral set X admits a complex Haar system only if X is embeddable in the plane. We replace the requirement that X be a finite polyhedral set with several weaker assumptions.

Let X be a compact Hausdorff space, and let $C(X)$ be the linear space of continuous complex valued functions on X. A subspace M of $C(X)$ of dimension $n \geqq 2$ is said to be a complex Haar system if and only if each nonzero member of M has at most $n-1$ zeros in X. Haar and Kolmogoroff (see [6, Theorem 19]) showed that Haar systems are precisely those finite-dimensional subspaces of $C(X)$ that permit a unique best Chebyshev approximation to each f in $C(X)$.

This article owes its being to Professor R. Creighton Buck who supervised its writing in my dissertation [7]. Credit is also due Professor Edward R. Fadell who made many useful suggestions.

2. **Main results.** By a k-ode we mean a homeomorph of the subspace of the plane consisting of k distinct radii of unit length drawn from the origin, and by a disk we mean a homeomorph of the closed unit disk. Also, we will say that X is of type H if and only if X is a compact connected Hausdorff space such that $C(X)$ contains a Haar system. Embeddable always means "in the plane."

In my dissertation I showed:

(A) *A space of type H that contains a disk is embeddable*; and

(B) *a locally connected space of type H that contains as an open set a k-ode for some $k \geqq 3$ is embeddable.*

Also, I conjectured:

(C) *Any locally connected space of type H is embeddable*; and

(D) *not every space of type H is embeddable.*

Since then, Professors Brian R. Ummel and George Henderson of the University of Wisconsin, Milwaukee, have verified (C).

In summary we now have

THEOREM. *Any space X of type H that is not embeddable is a* 1-

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dimensional continuum that is not locally connected.

3. **Other results.** Our proofs depend on a simple

LEMMA 1. *Let M be a Haar system on X, and let $f_1, \cdots, f_n$ be a basis for M. For $x_1, \cdots, x_n$ in X, define*

$$D(x_1, \cdots, x_n) = \det[f_i(x_j)], \qquad i, j = 1, \cdots, n,$$

and

$$\phi_i(x) = D(x, x_i, x_3, \cdots, x_n), \qquad i = 1, 2.$$

If $x_1, \cdots, x_n$ are distinct points of X, then $\phi = \phi_1/\phi_2$ is a one-to-one continuous transformation on $X - \{x_2, \cdots, x_n\}$ to the plane. It follows that ϕ is a homeomorphism from the complement in X of any open set containing $\{x_2, \cdots, x_n\}$, to the plane.

The next two lemmas are consequences of Lemma 1.

LEMMA 2. *If X admits a Haar system, then X is the union of two compact sets each of which is embeddable in the plane. It follows, for example, that X is a separable metric space whose dimension is at most two.*

LEMMA 3. *If X admits a Haar system and if X contains a point p such that $X - \{p\}$ is the union of separated sets X_1 and X_2 such that both $X_1 \cup \{p\}$ and $X_2 \cup \{p\}$ contain an arc containing p, then X is embeddable in the plane.*

4. **Comments.** (1) Mairhuber, in 1955, in his thesis under Schoenberg, showed that $C_R(X)$ contains a real Haar system if and only if X is a part of a circle.

(2) The results of §1 hold if the requirement that a space of type H be connected is dropped.

(3) To prove (A), we use Lemma 1, with the x's chosen from the interior U of the disk, to define a homeomorphism h from $X - U$ to the plane. Using the Jordan Curve Theorem, we extend h to a homeomorphism from all of X to S^2. Since Schoenberg, Yang, and Loewner [8] showed that S^2 is not of type H, $h[X]$ is a proper subset of S^2. That is, h is an embedding.

(4) To prove (B), we use Lemma 3 to reduce the problem to the case that the vertex p of a small k-ode, say K, is not a cut-point of X. Let U be the open k-ode contained in K having as endpoints the midpoints of the radii of K. Use Lemma 1 with the x's in U. We extend the resulting homeomorphism h to all of X by means of a

homotopy argument. Let u and v be endpoints of U. Slide the x's off the radii from p to u and v in such a way that at each stage $x_1, \cdots, x_n$ are distinct points of U. The resulting homeomorphisms eventually map u and v to the same component of the complement of the images of $X-K$. It follows that the same situation for $h(u)$, $h(v)$ and $h[X-K]$ holds. Hence, each of the endpoints of U are mapped into the same component of $h[X-K]$. A k-ode can be placed in this component in such a way as to extend h. To make the homotopy argument precise, we use the fact that $X-U$ is a locally connected continuum. (See [**9**, Theorem 2.41].)

(5) Lemma 3 coupled with the following result of Claytor (1935) shows that (C) is true.

THEOREM (CLAYTOR). *A locally connected continuum that contains no cut-points and that contains neither of the Kuratowski graphs K_1 and K_2 is embeddable in the sphere.*

A locally connected space of type H contains neither K_1 nor K_2 by (B). As in comment (3), the image is a proper subset of S^2, that is, a subset of the plane.

The relevance of Claytor's work to this problem was pointed out by Ummel and Henderson. They have an alternative approach to the result based on a later paper by Claytor (1937).

(6) We suggest the following replacement for (C).

Conjecture. An arcwise connected space of type H is embeddable.

(7) Regarding (D), consider the classical example of the "Lakes of Wada." (See [**3**, p. 143].) Here only one fresh water lake is needed instead of two. Connect a point of the shore of the sea with a point of the shore of the lake by an arc that touches the island at only those two points. The resulting space is a 1-dimensional continuum that is not embeddable but such that the complement of any open set is embeddable. We have not been able to show that this space is of type H. Incidentally, a space that satisfies (D) is necessarily a subspace of 3-space (see [**4**, Theorem V2]).

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UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706