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# THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR THE SPHERES

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The Adams spectral sequence has been an important tool in research on the stable homotopy of the spheres. In this note we outline new information about a variant of the Adams sequence which was introduced by Novikov [7]. We develop simplified techniques of computation which allow us to discover vanishing lines and periodicity near the edge of the $E_2$-term, interesting elements in $E_2^{2,*}$, and a counterexample to one of Novikov's conjectures. In this way we obtain independently the values of many low-dimensional stems up to group extension. The new methods stem from a deeper understanding of the Brown-Peterson cohomology theory, due largely to Quillen [8]; see also [4]. Details will appear elsewhere; or see [11].

When $p$ is odd, the $p$-primary part of the Novikov sequence behaves nicely in comparison with the ordinary Adams sequence. Computing the $E_2$-term seems to be as easy, and the Novikov sequence has many fewer nonzero differentials (in stems $\leqq 45$, at least, if $p=3$), and periodicity near the edge. The case $p=2$ is sharply different. Computing $E_2$ is more difficult. There are also hordes of nonzero differentials $d_3$, but they form a regular pattern, and no nonzero differentials outside the pattern have been found. Thus the diagram of $E_4$ ($=E_\infty$ in dimensions $\leqq 17$) suggests a vanishing line for $E_\infty$ much lower than that of $E_2$ of the classical Adams spectral sequence [3].

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**1. The spectral sequence.** The construction of the classical Adams spectral sequence for the spheres [1] works equally well if the spec-

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[1] Most of the results announced in this paper were in the author's doctoral dissertation, written under an NSF graduate fellowship at the University of Chicago.

trum $K(Z_p)$ representing ordinary cohomology is replaced by an arbitrary ring spectrum $X$. If $X$ satisfies certain conditions, the $E_2$-term of the resulting sequence will be isomorphic to

$$\mathrm{Ext}_{A^X}(\Lambda^X, \Lambda^X),$$

where $A^X = X^*(X)$ is the algebra of operations in $X$-cohomology theory and $\Lambda^X = \pi_*(X)$ is the coefficient ring. Novikov showed [7] that if $X = MU$ (the spectrum representing complex cobordism) this multiplicative spectral sequence converges to the stable homotopy ring $\pi_*^S$:

$$E_\infty^{s,t} \cong F^s\pi_{t-s}^S / F^{s+1}\pi_{t-s}^S,$$

where $F^*$ is a filtration of $\pi_*^S$. Furthermore, if $X' = BP_p$, the Brown-Peterson spectrum [4] for the prime $p$, the resulting spectral sequence $\{{}_pE_r, {}_pd_r\}$ is exactly the $p$-primary part $\{E_r \otimes Q_p, d_r \otimes Q_p\}$ of the $MU$ spectral sequence ($Q_p$ is the ring of rational numbers with denominators prime to $p$.)

Not much is known about the $MU$ spectral sequence, because even limited computations of $E_2$ have been difficult. This is regrettable, since what *is* known indicates that the Novikov sequence has certain a priori advantages over the usual one. The nonzero terms are sparse, for example: ${}_pE_2^{s,t} = 0$ if $t \not\equiv 0 \bmod 2(p-1)$. Furthermore, almost all of the image of the $J$-homomorphism [2], [9] lies on the line $s = 1$, in the following sense. According to Novikov, $E_2^{1,2t} = Z_{m(t)}\langle \alpha_t \rangle$, a cyclic group with generator $\alpha_t$, isomorphic to the image of $J$ in dimension $2t-1$ (isomorphic to $Z_2$ if $2t-1 \equiv 5 \bmod 8$). There is a map $q_1 : \pi_n^S \to E_2^{1,n+1}$ such that an element of $E_2^{1,n+1}$ survives to $E_\infty$ iff it belongs to im $q_1$. Furthermore, if $\tilde{q}_1$ denotes the restriction of $q_1$ to im $J$, then [7, Chapters 10 and 11]

(1) if $n = 8k+1$, $E_\infty^{1,n+1} = E_\infty^{1,n+1} = Z_2$;

(2) if $n = 8k+3$ ($k > 0$), then im $q_1 =$ im $\tilde{q}_1$ has index 2 in $E_2^{1,n+1} = Z_{m(4k+2)}$, and $\tilde{q}_1$ has kernel $Z_2$; in fact, $d_3\alpha_{4k+2} = h^3\alpha_{4k} \neq 0$;

(3) if $n = 8k+5$, $E_2^{1,n+1} = Z_2$ does not survive to $E_\infty$; in fact, $d_3\alpha_{4k+3} = h^3\alpha_{4k+1} \neq 0$;

(4) if $n = 8k+7$, im $\tilde{q}_1 = Z_{m(4k+4)} = E_2^{1,n+1} = E_\infty^{1,n+1}$.

Here $h = \alpha_1$.

2. **Quillen's algebra.** Novikov knew that, given a prime $p$, the algebra $A^{BP} = BP^*(BP)$ was much simpler than $A^{MU} \otimes Q_p$, but he did not have complete information about $A^{BP}$. Later, Quillen [8] discovered an idempotent $\epsilon$, which split the spectrum $MUQ_p$ into a sum of suspensions of the spectrum $BP_p$ [4]. Now

$$\pi_*(BP) = Q_p[k_1, k_2, \cdots], \qquad H_*(BP) = Q_p[m_1, m_2, \cdots],$$

with $|k_i| = -|m_i| = -2(p^i-1)$. We can take $m_i = (1/p^i)h\epsilon[CP^{p^i-1}]$; the Hurewicz homomorphism $h$ is monic, and may be computed using Quillen's formal-group techniques [11] or standard methods. Thanks to the idempotent $\epsilon$, Quillen and Adams were able to write down explicit formulas for the Hopf-algebra structure of the algebra of operations $A^{BP}$ ($=A$, for short).

First, there is a coalgebra $R$ of operations, free as a $Q_p$-module on generators $r_E$, where $E$ runs over all finitely nonzero sequences $(e_1, e_2, \cdots)$ of nonnegative integers and $|r_E| = 2(\sum (p^i-1)e_i)$. The diagonal map is given by $\phi^* r_E = \sum_{E'+E''=E} r_{E'} \otimes r_{E''}$. Then $\Lambda' = \pi_*(BP)$ is an algebra over the coalgebra $R$, with action given (via the Hurewicz map) by $r_E m_n = m_{n-i}$ if $e_i = p^{n-i}$ and all other $e_j$ are zero, and $r_E m_n = 0$ otherwise. Moreover, multiplication by an element $\lambda$ of $\Lambda'$ is also a $BP$-cohomology operation, and in fact every operation can be written as a (possibly infinite) sum $\sum \lambda_i r_{E_i}$ in which the degree of each $\lambda_i r_{E_i}$ is a constant independent of $i$. Unfortunately, the composition $r_E r_F$ of two operations in $R$ does not usually lie in $R$; however, it can be written uniquely as a finite sum $r_E r_F = \sum_K c_K r_K$ with $c_K \in \Lambda'$, using the methods of [11] or those of [4]. This enables us to express compositions $(\lambda r_E)(\lambda' r_F)$ in the form $\sum \lambda_i r_{E_i}$. Thus the algebra $A$ of all operations is the completed tensor product $\Lambda' \hat{\otimes} R$.

Proposition 1. *Let $\bar{\Lambda}$ be the two-sided ideal in $A$ generated by all elements of $\Lambda$ of negative degree. Let $\mathcal{A}_p/(Q_0)$ be the algebra of reduced Steenrod pth powers* [6]. *Then there is an isomorphism $f: A/\bar{\Lambda} \cong \mathcal{A}_p/(Q_0)$.*

Proof. Let $\mathrm{Th}: BP_p \to K(Z_p)$ be the $Z_p$ Thom class. Then

$$\begin{array}{ccc} \bar{f} = \mathrm{Th}_*: [BP, BP] & \to & [BP, K(Z_p)] \\ \| & & \| \\ A & & H^*(BP; Z_p) \\ & & \| \\ & & \mathcal{A}_p/(Q_0) \end{array}$$

satisfies

$$\begin{aligned} \bar{f}(k^E r_F) &= c(\mathcal{P}^F), \qquad && E = 0 \ [6]; \\ &= 0, && \text{otherwise}; \end{aligned}$$

where $c$ is the canonical antiautomorphism. The map $\bar{f}$ induces the required $f$ on $A/\bar{\Lambda}$.

A generator $r_E$ is *indecomposable* if it cannot be expressed as a finite sum $r_E = \sum \lambda_i R_i R_i'$, where $\lambda_i \in \Lambda'$; $R_i$, $R_i' \in R$; and $|R_i|$, $|R_i'| > 0$.

THEOREM 2. *The generator $r_E$ of $R$ is indecomposable if and only if $E = (p^i, 0, 0, \cdots)$, $i \geqq 0$. Moreover, $pr_{(p^i,0,0,\ldots)}$ is decomposable.*

The proof is obtained by noticing certain pleasant properties of the multiplication table for $R$ and applying them in the proper sequence.

3. **Resolutions over $A$.** To compute Ext we must construct resolutions over $A$, which seems difficult at first glance since $R$ is not an algebra, $A$ is not connected, and the ground ring $Q_p$ is not a field. The next proposition shows how to circumvent some of these difficulties. Define the filtrations $F^s\Lambda' = \sum_{i \leq 2s} (\Lambda')^i$, $F^sA = F^s\Lambda' \hat{\otimes} A$, and $F^sM = (F^sA)M$ if $M$ is an $A$-module. We have

$$0 \to F^1M \xrightarrow{i} M \xrightarrow{j} \operatorname{cok} i \to 0.$$

Write $JM$ for cok $i$; then $J$ is easily made into a functor on the category of $A$-modules.

PROPOSITION 3. *There exist complexes*

$$C\colon \cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 = A \to \Lambda' \to 0$$

*satisfying*

(1) $C_1 = \sum Au_j$ *with* $d_1u_j = r_{(p^j,0,0,\ldots)}$;

(2) $C_i = \prod_j Aw_j^{(i)}$ *is locally finitely generated as an $A$-module*, $i > 1$;

(3) $\ker(Jd_i) \subset j(\operatorname{im} d_{i+1})$ *in* $JC_i$ *for all* $i$, $n \geqq 0$.

*Any such $C$ is an $A$-projective resolution of $\Lambda'$.*

The proof is straightforward. Notice that the infinite direct product $\prod Aw_j^{(i)}$ is not necessarily free over $A$; it is projective, however. As a further aid to computation there is

LEMMA 4. *If $\{C_i, d_i\}$ is any $A$-projective resolution of $\Lambda'$, write $C_i^* = \operatorname{Hom}_A^*(C_i, \Lambda')$, $d_i^* = \operatorname{Hom}_A^*(d_i, \Lambda')$. Then*

$$\operatorname{Ext}_A^{s,t}(\Lambda', \Lambda') = \operatorname{Tors}(\operatorname{cok}(d_s^*)^t), \qquad (s, t) \neq (0, 0).$$

PROOF. This follows from the fact that $\operatorname{Ext}_{A'}^{s,t}$ is finite for $(s, t) \neq (0, 0)$ [7, Corollary 2.1].

Thus in determining Ext we need know just the boundaries, and not the cycles too. In fact we can even work over $Z_{p^f}$ for suitable $f$.

Now we can prove

PROPOSITION 5. $\mathrm{Ext}^{0,t}=0$ *unless* $t=0$; $\mathrm{Ext}^{0,0}=Z$.

THEOREM 6. $\mathrm{Ext}^{2,t}$ *contains a direct summand isomorphic to* $Z_p$ *for* $t=2p^i(p-1)$ $(i\geqq 1)$ *and* $t=2(p^i+1)(p-1)$ $(i>1)$.

THEOREM 7. *For* $p=2$, *the element of* $\mathrm{Ext}^{2,2^i}$ *found in Theorem* 6 *maps to the Arf-invariant element* $h_i^2$ *of the classical Adams spectral sequence* [5].

PROOF. Apply the Thom map (Proposition 1) to a suitable $A$-resolution.

PROPOSITION 8. *The two-primary part* ${}_2\mathrm{Ext}^{s,t}$ *has the following "edge" values*:

$$
\begin{aligned}
{}_2\mathrm{Ext}^{n,2(n+k)} &= 0, && k<0;\\
&= Z_2, && k=0,\ n\geqq 1 \ \text{(generated by } h^n);\\
&= 0, && k=1,\ n\geqq 2;\\
&= Z_2, && 2\leqq k\leqq 5,\ n\geqq 4 \ \text{(generated by } h^{n-1}\alpha_{k+1}).
\end{aligned}
$$

Further computations of the additive structure of ${}_2\mathrm{Ext}^{*,*}$ in low dimensions are given in Figure 1. Thanks to Proposition 8, the first three nonzero Novikov differentials $d_3\alpha_i=h^3\alpha_{i-1}$, $i=3, 6, 7$, give rise to infinite towers of nonzero $d_3$'s. Moreover, every other differential in the range $t-s\leqq 17$ must be zero for dimensional reasons. Finally, ${}_2E_\infty$ has a vanishing line considerably lower than that of the $E_\infty$-term of the classical Adams spectral sequence in this range of dimensions. We conjecture that the preceding four sentences are also true without restriction on the dimensions.

Similar computations for $p=3$ disclose striking edge properties like Proposition 8, but many fewer differentials. Contrary to Novikov's conjecture [7], there is a nonzero differential $d_5:E_2^{2,36}\rightarrow E_2^{7,40}$ for $p=3$. This differential, whose existence is inferred from Toda's result [10], also gives rise to an infinite family of nonzero differentials. It is encouraging that there is only one nonzero differential in the range $t-s\leqq 40$, as compared to 17 in the classical 3-primary Adams spectral sequence.

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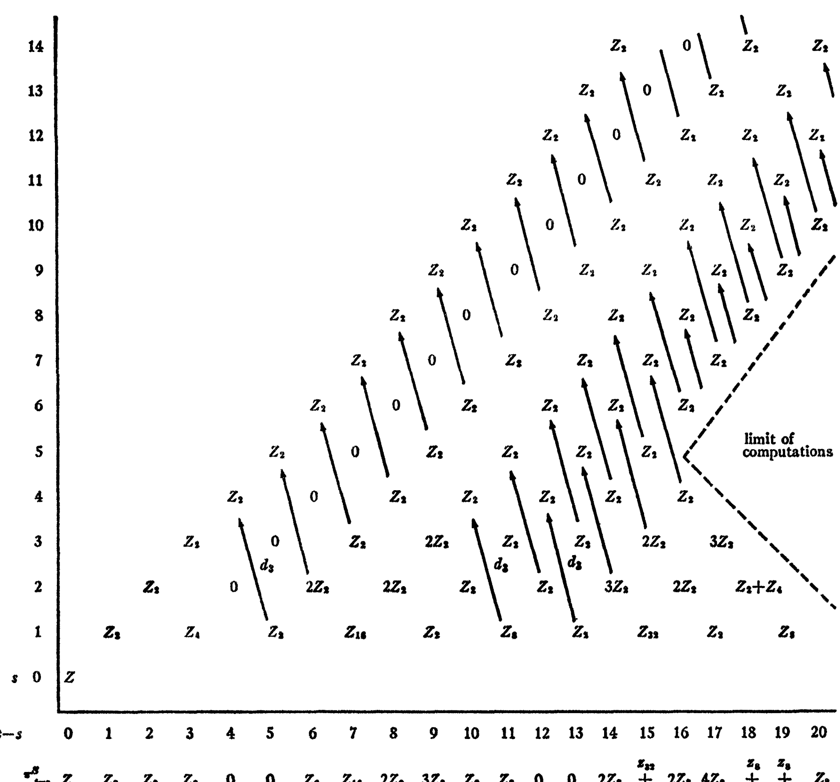


FIGURE 1. ${}_2\mathrm{Ext}^{s,t}$ for the Novikov sequence.