BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, January 1971

ON THE DEMIREGULARITY OF WEAK SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

BY JINDŘICH NEČAS

Communicated by Felix Browder, June 29, 1970

1. **Introduction.** Let Ω be a bounded domain with infinitely differentiable boundary $\partial\Omega$ in n-dimensional real space R_n. Let k be a positive integer, and let us define the functions $a_i(x, \xi)$ for multi-indices $|i| = i_1 + i_2 + \cdots + i_n \leqq k$, continuous in $\bar{\Omega} \times R_\kappa$, where κ is the number of indices of length $\leqq k$. By $W_p^{(k)}(\Omega)$, we denote the Sobolev space of L_p-functions whose derivatives up to the order k are also L_p-functions, with the norm

$$\|u\|_{k,p} = \left(\int_\Omega \sum_{|i| \leq k} | D^i u |^p dx \right)^{1/p},$$

where the usual notation

$$D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

is introduced. The functions $a_i(x, \xi)$ are supposed to satisfy the growth-conditions:

(1.1) $|a_i(x, \xi)| \leqq c(1 + |\xi|)$.

Let functions $u_0 \in W_2^{(k)}(\Omega)$ and $f_i \in L_2(\Omega)$, $|i| \leqq k$, be given. Let $\mathring{W}_p^{(k)}(\Omega)$ be the closure of $D(\Omega)$, the space of infinitely differentiable functions with compact support, in the space $W_p^{(k)}(\Omega)$.

A function u from $W_2^{(k)}(\Omega)$ is called a weak solution of the Dirichlet problem: $\partial^l u/\partial n^l = \partial^l u_0/\partial n^l$ on $\partial\Omega$, $l = 0, 1, \cdots, k-1$, (where $\partial/\partial n$ is the derivative with respect to the outer normal),

$$\sum_{|i| \leq k} (-1)^{|i|} D^i(a_i(x, \xi(u))) = \sum_{|i| \leq k} (-1)^{|i|} D^i f_i \qquad \text{in } \Omega$$

(where the components of $\xi(u)$ are $D^i u$) if

(1.2) $u - u_0 \in \mathring{W}_2^{(k)}(\Omega)$,

(1.3) for every v in $\mathring{W}_2^{(k)}(\Omega)$:

$$\int_\Omega \sum_{|i| \leq k} D^i v a_i(x, \xi(u)) dx = \int_\Omega \sum_{|i| \leq k} D^i v f_i dx.$$

Copyright © 1971, American Mathematical Society

We will suppose the following:

$$\sum_{|i|\leq k} a_i(x,\xi)\xi_i \geqq c_1 \sum_{|i|\leq k} \xi_i^2 - c_2. \tag{1.4}$$

For the sake of simplicity, we suppose the differentiability of $a_i(x, \xi)$ with respect to ξ_j and

$$\left|\frac{\partial a_i}{\partial \xi_j}\right| \leqq c, \qquad \sum_{|i|,|j|\leq k} \frac{\partial a_i}{\partial \xi_j}\eta_i\eta_j \geqq c \sum_{|i|=k} \eta_i^2. \tag{1.5}$$

The following condition for asymptotic behaviour of $a_i(x, \xi)$ is required: there exists continuous $a_{ij}(x)$ in $\bar{\Omega}$, $|i|$, $|j| \leqq k$, such that

$$\sum_{|i|,|j|\leq k} a_{ij}(x)\xi_i\xi_j \geqq c_1 \sum_{|i|=k} \xi_i^2 \tag{1.6}$$

and such that for $t>0$:

$$\left|\frac{a_i(x,t\xi)}{t} - \sum_{|j|\leq k} a_{ij}(x)\xi_j\right| \leqq c(t)(1 + |\xi|), \tag{1.7}$$

where $c(t)\rightarrow 0$ for $t\rightarrow\infty$.

The main result is:

THEOREM. *Let* $2\leqq p<\infty$ *and* $u_0\in W_p^{(k)}(\Omega)$, $f_i\in L_p(\Omega)$. *Let the conditions* (1.1), (1.4)—(1.7) *be satisfied. Then there exists a unique weak solution of the Dirichlet problem belonging to the space* $W_p^{(k)}(\Omega)$. *It satisfies the inequality*:

$$\|u\|_{k,p} \leqq c(p)\left(1 + \sum_{|i|\leq k} \|f_i\|_{0,p} + \|u_0\|_{k,p}\right). \tag{1.8}$$

It is well known that the regularity problem consists of proving that the weak solution belongs to the class $C^{(k),\mu}$, the class of functions whose derivatives up to order k are μ-Hölder continuous (in Ω or $\bar{\Omega}$). The solution of this problem is not known in general. Under certain conditions, given more general growth of the functions $a_i(x,\xi)$: $|a_i(x,\xi)| \leqq c(1+|\xi|^{m-1})$, $1<m<\infty$, the answer is affirmative for the case of one second-order equation; see, for example, O. A. Ladyženskaja-N. N. Uralceva [5], Ch. B. Morrey [7], and for $n=2$, $k\geqq 1$, see J. Nečas [9]. For higher dimensions and order, or for systems of second or higher order, this problem is still open. There is a counterexample under a slightly different hypothesis for the second-order systems of E. Giusti and M. Miranda [4], where the solution is bounded, but not continuous. This situation implies the definition of partial regularity: there exists a set F closed in Ω with $\text{mes}(F)=0$, such that the weak solution belongs to $C^{(k),\mu}(\Omega\backslash F)$.

Partial regularity was proved in the papers of Ch. B. Morrey [8], E. Giusti-M. Miranda [3], E. Giusti [2]. If we look to the scale $W_p^{(k)}$, $2 \leqq p \leqq \infty$, and if we extend it further to $C^{(k),\mu}$ for $0<\mu<1$, we see that the cut between weak and regular solutions is the space $W_\infty^{(k)}$. Hence, a weak solution is called demiregular if $u \in \bigcap_{p\geq 2} W_p^{(k)}(\Omega)$, and this is an immediate consequence of our theorem, provided $u_0 \in W_\infty^{(k)}(\Omega)$ and $f_i \in L_\infty(\Omega)$.

2. **Proof of the Theorem.** We use the following nontrivial lemma from the theory of linear elliptic equations, see, for example, J. L. Lions, E. Magenes [6].

LEMMA 1. *Let w be a weak solution of*

$$\sum_{|i|,|j|\leq k} (-1)^{|i|} D^i(a_{ij}(x) D^j w) = \sum_{|i|\leq k} (-1)^{|i|} D^i f_i$$

in Ω, with $f_i \in L_p(\Omega)$, $\infty > p > 1$, $w - u_0 \in \mathring{W}_p^{(k)}(\Omega)$, $u_0 \in W_p^{(k)}(\Omega)$ and with a_{ij} satisfying (1.6). *Then there exists a unique solution and*

$$\|w\|_{k,p} \leqq \Big(\sum_{|i|\leq k} \|f_i\|_{0,p} + \|u_0\|_{k,p} \Big). \tag{2.1}$$

As an immediate consequence of Lemma 1, we obtain:

LEMMA 2. *For $w \in \mathring{W}_p^{(k)}(\Omega)$, $\infty > p \geqq 2$,*

$$\sup_{\|v\|_{k,p'} \leq 1, v \in \mathring{W}_{p'}^{(k)}(\Omega)} \int_\Omega \sum_{|i|,|j|\leq k} a_{ij}(x) D^i v D^j w \, dx \geqq c(p) \|w\|_{k,p},$$

where $1/p' + 1/p = 1$.

Using well-known results about monotone operators, their applications to nonlinear boundary value problems, compare F. E. Browder [1], we have:

LEMMA 3. *Under the conditions* (1.1), (1.4), (1.5), *there exists a unique solution of* (1.2), (1.3) *and*

$$\|u\|_{k,2} \leqq c\Big(1 + \sum_{|i|\leq k} \|f_i\|_{0,2} + \|u_0\|_{k,2}\Big). \tag{2.2}$$

Proof of the theorem. Let $0 \leqq \tau \leqq 1$, and let us consider the family of differential operators defined as

$$(1-\tau) \sum_{|i|,|j|\leq k} (-1)^{|i|} D^i(a_{ij}(x) D^j u) + \tau \sum_{|i|\leq k} (-1)^{|i|} D^i(a_i(x, \xi(u))). \tag{2.3}$$

We can easily see that the conditions (1.1), (1.4), (1.5) and (1.7) are valid with constants independent of τ. Hence, for $0 \leqq \tau \leqq 1$, there exists a unique solution of our problem in $W_2^{(k)}(\Omega)$. For $\tau=0$, we have, in virtue of Lemma 1, the assertion of the theorem.

(i) Let the assertion be valid for some τ_0. Then it is true for $\tau_0 \leqq \tau < \tau_0+\epsilon \leqq 1$ with some $\epsilon>0$.

Let $v \in W_p^{(k)}(\Omega)$ be such that $v-u_0 \in \mathring{W}_q^{(k)}(\Omega)$ and let us define the operator $A: v \rightarrow Av$ such that Av is the solution of the problem with the functions

$$f_i + (\tau - \tau_0) \sum_{|j| \leq k} a_{ij}(x) D^j v + (\tau_0 - \tau) a_i(x, \xi(v))$$

substituted for f_i.

For $\tau=\tau_0$, we obtain from (1.8) that the solution belongs to the ball $\|u\|_{k,p} \leqq R$ where

$$R = (c(p) + 1)\left(1 + \sum_{|i| \leq k} \|f_i\|_{0,p} + \|u_0\|_{k,p}\right).$$

Let us take first v in the ball $\|v\|_{k,p} \leqq 2R$ and then ϵ small enough such that $\|Av\|_{k,p} \leqq 2R$. It follows from (1.5):

$$\int_\Omega \sum_{|i| \leq k} (a_i(x, \xi(u_1)) - a_i(x, \xi(u_2))) D^i(u_1 - u_2) dx \geqq c \sum_{|i| \leq k} \|D^i(u_1 - u_2)\|_{0,2}^2.$$

Hence, with ϵ small enough

$$\|A(v_1) - A(v_2)\|_{k,2} \leqq \alpha \|v_1 - v_2\|_{k,2}, \qquad 0 \leqq \alpha < 1. \tag{2.4}$$

If we introduce into the set $\|v\|_{k,p} \leqq 2R$, $v-u_0 \in \mathring{W}_p^{(k)}(\Omega)$, the metric induced by the norm $\|v\|_{k,2}$, we obtain a complete metric space and the operator A is a contraction. This implies the existence of a fixed point, which is a solution of (1.2), (1.3) belonging to $W_p^{(k)}(\Omega)$. From (1.8), this estimation for $\tau_0 \leqq \tau < \tau + \epsilon$ with ϵ small enough follows.

(ii) For $0 \leqq \tau \leqq 1$ and $u \in W_p^{(k)}$ the solution of the problem, an estimation (1.8) holds with $c(p)$ independent of τ. Let us suppose the contrary. Then for n integers, there exists τ_n and $f_i^n \in L_p$, $u_0^n \in W_p^{(k)}$, with $u_n \in W_p^{(k)}$ the solutions of the problem, such that

$$\|u_n\|_{k,p} \geqq n\left(1 + \sum_{|i| \leq k} \|f_i^n\|_{0,p} + \|u_0^n\|_{k,p}\right).$$

Let

$$t_n = \|u_n\|_{k,p}, \qquad v_n = u_n / t_n.$$

If we put $g_i^u = (1/t_n) f_i^n$ and $v_0^n = u_0^n / t_n$, we obtain that $g_i^n \rightarrow 0$ in L_p and $v_0^n \rightarrow 0$ in $W_p^{(k)}$. We have for $\varphi \in \mathring{W}_{p'}^{(k)}$:

$$(1-\tau_n)\int_\Omega \sum_{|i|,|j|\le k} a_{ij}(x) D^i\varphi D^j v_n dx + \tau_n \int_\Omega \sum_{|i|\le k} \frac{1}{t_n} a_i(x, t_n\xi(v_n)) D^i\varphi dx$$

$$= \int_\Omega \sum_{|i|\le k} D^i \varphi \overset{n}{g}_i dx$$

$$= \int_\Omega \sum_{|i|,|j|\le k} a_{ij}(x) D^i\varphi D^j v_n dx$$

$$+ \tau_n \int_\Omega \sum_{|i|\le k} \left(\frac{1}{t_n} a_i(x, t_n\xi(v_n)) - \sum_{|j|\le k} a_{ij}(x) D^j v_n\right) D^i\varphi dx.$$

In virtue of Lemma 2, we can choose φ_n such that $\|\varphi_n\|_{k,p'}=1$ and

$$\int_\Omega \sum_{|i|,|j|\le k} a_{ij}(x) D^i\varphi_n D^j (v_n - \overset{n}{v}_0) dx \ge c_1 > 0 \qquad \text{for } n \ge n_0,$$

which implies for $n \ge n_0'$:

$$\int_\Omega \sum_{|i|,|j|\le k} a_{ij}(x) D^i\varphi_n D^j v_n dx \ge c_2 > 0. \tag{2.5}$$

Because of (1.7), we obtain

$$\lim_{n\to\infty} \left| \int_\Omega \sum_{|i|\le k} \left(\frac{1}{t_n} a_i(x, t_n\xi(v_n)) - \sum_{|j|\le k} a_{ij}(x) D^j v_n\right) \cdot D^i\varphi_n dx \right|$$

$$\le c(t_n)(\|\varphi_n\|_{k,1} + \|\varphi_n\|_{k,p'}\|v_n\|_{k,p}) \to 0$$

which gives, together with (2.5) and because $g_i^n \to 0$ in L_p, the contradiction.

(iii) By standard argument, the set S of τ where the theorem is valid, is closed; this follows from the fact that if $\tau_n \in S$ and u_n are solutions, then as above, $u_n \to u$ in $W_2^{(k)}$ where u is the solution for $\tau = \lim_{n\to\infty} \tau_n$. But, since

$$\|u_n\|_{k,p} \le c\left(1 + \sum_{|i|\le k} \|f_i\|_{0,p} + \|u_0\|_{k,p}\right),$$

the same is true for u. As in (ii), the set S is open; so it is the whole interval $\langle 0, 1\rangle$. q.e.d.

Bibliography

1. F. E. Browder, *Existence and uniqueness theorems for solutions of non-linear boundary value problems*, Proc. Sympos. Appl. Math., vol. 17, Amer. Math. Soc., Providence, R. I., 1965, pp. 24–49. MR **33** #6092.

2. E. Giusti, *Regolarità parziale delle soluzioni di sistemi ellitici quasi-lineari di ordine arbitrario*, Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 115–141. MR **40** #527.

3. E. Giusti and M. Miranda, *Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari*, Arch. Rational Mech. Anal. **31** (1968/69), 173–184. MR **38** #3574.

4. ———, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Un. Mat. Ital. (4) **1** (1968), 219–226. MR **38** #591.

5. O. A. Ladyženskaja and N. N. Ural'ceva, *Linear and quasi-linear equations of elliptic type*, "Nauka", Moscow, 1964; English transl., Academic Press, New York, 1968. MR **35** #1955.

6. J.-L. Lions and E. Magenes, *Problemi ai limiti non omogenei*. III, V, Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 41–103; ibid. (3) **16** (1962), 1–44. MR **26** #4048; 4049.

7. Ch. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der math. Wissenschaften, Band 130, Springer-Verlag, New York, 1966. MR **34** #2380.

8. ———, *Partial regularity results for non-linear elliptic systems*, J. Math. Mech. **17** (1967/68), 649–670. MR **38** #6224.

9. J. Nečas, *Sur la régularité des solutions faibles des équations elliptiques non linéarires*, Comment. Math. Univ. Carolinae **9** (1968), 365–413. MR **39** #3142.

UNIVERSITY OF ILLINOIS, CHICAGO, ILLINOIS 60680