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CURVATURE AND DIFFERENTIABLE STRUCTURE ON SPHERES[1]

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1. **Introduction.** The purpose of this note is to outline a proof of the following result: A simply connected, complete, riemannian manifold whose curvature tensor R is sufficiently close to the curvature tensor R_0 of the standard sphere S of the same dimension is diffeomorphic to S. Traditionally, the proximity of R and R_0 has been measured in terms of the sectional curvature as follows: A riemannian manifold is called δ-pinched if the sectional curvature K satisfies the condition $\delta < K \leqq 1$. Using this concept, Gromoll [4] and Calabi proved the following diffeomorphism theorem: There exists a sequence δ_n with $\lim \delta_n = 1$ as n increases such that a δ_n-pinched simply connected riemannian manifold M of dimension n is diffeomorphic to the sphere S^n.

In order to express the main condition of the diffeomorphism theorem independently of dimension, we introduce a different measurement for the proximity of the curvature tensors R and R_0 of the manifolds M and S^n respectively. To formulate this condition we think of the riemannian curvature tensor as a selfadjoint, linear map $R: V\wedge V \to V\wedge V$, where $V\wedge V$ denotes the exterior product of the tangent space with itself. A riemannian manifold is called *strongly δ-pinched*, if the eigenvalues λ of the above linear map at every point of M satisfy the condition $\delta < \lambda \leqq 1$.

2. **Statement of result.** In previous studies the pinching constant depended on the dimension of the manifold. However, the introduction of strong δ-pinching has the following advantage: The constant δ in the theorem below is independent of the dimension of the manifold.

THEOREM. *There exists a constant $\delta \neq 1$ such that a complete, simply connected, strongly δ-pinched riemannian manifold is diffeomorphic to the standard sphere of the same dimension.*

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The main idea of the following proof is new. However, methods similar to those employed by Rauch [7], Berger [1], [2], Klingenberg [5], [6], Gromoll [4], and Cheeger [3] have been adapted to obtain the necessary estimates.

3. **Outline of proof.** We can suggest an idea of the proof by observing the Gauss map $g: M \to S^n$ that exists in case M is an n-dimensional manifold embedded in euclidian space E^{n+1}. Of course, the map g sending $x \in M$ into the unit normal vector at x translated to a fixed point x_0 is well defined because parallel translation in $E = M \times E^{n+1} = \tau(M) \oplus \nu(M)$, where $\tau(M)$ and $\nu(M)$ denote tangent and normal bundle respectively, is independent of the path. In addition, g is a local diffeomorphism as long as the derivative $D_X n$ of the unit normal vector field n in any direction $X \neq 0$ is nonzero.

In the general case the normal bundle is not available; however, we replace it by a trivial line bundle ϵ and define a flat connection ∇' on $E = \tau(M) \oplus \epsilon$. At this point a map $f: M \to S^n$ is defined by replacing the normal vector field by a section e of length one in ϵ; i.e., the image $f(x)$ is obtained by parallel translation of $e(x)$ to the fibre E^{n+1} over a fixed point x_0. Again, f is a local, and since M is simply connected, a global diffeomorphism as long as $\nabla'_X e \neq 0$. Therefore, the proof consists of defining a flat connection ∇' on $\tau(M) \oplus \epsilon$ and checking $\nabla'_X e \neq 0$.

The first step in the construction of ∇' is to define a connection ∇'' in E with small curvature as follows:

$$\nabla''_X e_i = \nabla_X e_i - \tfrac{1}{2}(1 + \delta)\langle X, e_i \rangle e, \qquad \nabla''_X e = \tfrac{1}{2}(1 + \delta) X,$$

where ∇ denotes the riemannian connection in the tangent bundle $\tau(M)$; e_i, $i = 1, 2, \cdots, n$, denotes a moving orthonormal frame in $\tau(M)$; and e is a section of length one in ϵ. The curvature of ∇'' can be estimated in terms of δ. The idea for the definition of ∇'' originates from the following observation: In case M is the standard sphere embedded in E^{n+1}, the covariant derivative defined above is nothing but the ordinary derivative in E^{n+1}.

In the next step, ∇'' is used to construct a cross section u' in the principal bundle of $n+1$-frames associated to E. The results necessary for this construction are compiled in the first four chapters of [4]. The proofs are based on the Alexandrov-Rauch-Toponogoff comparison theorem and the Morse critical point theory. In particular, we use the representation of M as the union $M_0 \cup M_1$ of two balls representing upper and lower hemisphere. On M_0 we define a cross section u_0 by moving a fixed $n+1$-frame $u_0(q_0)$ chosen over the center q_0 of M_0 by parallel translation with respect to ∇'' along geodesic rays to points in M_0. On M_1 we define first $u_1(q_1)$ by parallel translation of $u_0(q_0)$ along a shortest geodesic to q_1, the center of M_1. Subsequently, u_1 is defined on M_1 by translation along geodesic rays. On $C = M_0 \cap M_1$ the sections u_0 and u_1 may not coincide, but the distance in the fibre can be estimated in terms of the pinching constant δ. Therefore, for δ close enough to 1, the sections u_0 and u_1 can be modified to yield a differentiable cross section u' on M. At this point, let ∇' denote the flat covariant derivative in $E = \tau(M) \oplus \epsilon$ corresponding to u'.

It remains to be shown that $\nabla'_X e \neq 0$. The result follows because for δ close to 1, the difference of ∇' and ∇'' is small and

$$\|\nabla''_X e\| = \tfrac{1}{2}(1 + \delta)\|X\| \sim \|X\|.$$

The details, as well as an estimate for the pinching constant δ, will be furnished in a subsequent paper.

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