THE HOMOLOGY OF THE CLASSICAL GROUPS OVER THE DYER-LASHOF ALGEBRA[1,2]

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1. **Introduction.** By Bott periodicity the classical groups, their classifying spaces, their homogeneous spaces, Im J and B Im J are infinite loop spaces, and hence the Dyer-Lashof operations act on their mod p homology for p a prime number. We will list the basic properties of the Dyer-Lashof operations in §2. These operations have been calculated in the homology of all these spaces, and the dual operations have also been computed. As applications one can calculate the indecomposable elements of the homology of these spaces over the Dyer-Lashof algebra R and the AR-Hopf algebra maps between any two classical groups or any two of their classifying spaces. In this paper we will summarize our results for BU, BU (the connected and infinite component versions of the classifying space of the infinite unitary group), O, SO, B Im J and Im J.

These results have been applied to the study of $H_*(F)$ and $H_*(BF)$ by J. P. May [9] and I. Madsen [7]. They are also useful in cobordism theory (see for example T. tom Dieck [5, p. 396]).

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NOTATION. All elements of a graded object will be indexed by their degree with the exception of the Chern and Wu classes. All homology and cohomology will have Z_p coefficients for p a prime number. When a result differs for p an odd prime and $p=2$ then the result for $p=2$ will be placed in square brackets.

2. **The Dyer-Lashof operations.** The homology of an infinite loop space B has natural homomorphisms $Q^i: H_*(B) \rightarrow H_*(B)$ for $i \geq 0$ of degree $2i(p-1)$ [of degree i] which have been studied by S. Araki and

T. Kudo [3], W. Browder [4], E. Dyer and R. Lashof [6], G. Nishida [10] and J. P. May [8]. Theorem 1 summarizes some of their results.

THEOREM 1. *The Dyer-Lashof operations satisfy the following properties*:

(a) $Q^0(\phi) = \phi$ and $Q^i(\phi) = 0$ if $i > 0$ where $\phi \in H_0(B)$ is the identity element for the loop product in $H_*(B)$.

(b) $Q^i(x) = 0$ if $2i < \deg x$ $[if\ i < \deg x]$.

(c) $Q^i(x) = x^p$ if $2i = \deg x$ $[if\ i = \deg x]$.

(d) $\sigma_* \circ Q^i = Q^i \circ \sigma_*$ where $\sigma_* : IH_*(\Omega B) \rightarrow H_*(B)$ is the homology suspension map.

(e) (*Multiplicative Cartan formula*) $Q^r(xy) = \sum_{i=0}^r Q^i(x) Q^{r-i}(y)$.

(f) (*Comultiplicative Cartan formula*)

$$\psi \circ Q^r(x) = \sum_{i=0}^r \sum Q^i(x') \otimes Q^{r-i}(x'')$$

where $\psi(x) = \sum x' \otimes x''$.

(g) $\chi \circ Q^i = Q^i \circ \chi$ where χ is the conjugation of $H_*(B)$.

(h) (*Nishida relations*) Let $P_*^s : H_*(B) \rightarrow H_*(B)$ of degree $-2s(p-1)$ $[of\ degree\ -s]$ be dual to the Steenrod operation P^s. Then

$$P_*^s \circ Q^r = \sum (-1)^{i+s} (s - pi, r(p-1) - ps + pi) Q^{r-s+i} \circ P_*^i.$$

We use the convention $(i, j) = (i+j)!/i!j!$ *if* $i \geq 0$ *and* $j \geq 0$ *while* $(i, j) = 0$ *if* $i < 0$ *or* $j < 0$. *Thus the above sum is taken over all integers* i.

(i) (*Adem relations*) *If* $a > pb$ *then*

$$Q^a \circ Q^b = \sum (-1)^{a+i} (pi - a, a - (p-1)b - i - 1) Q^{a+b-i} \circ Q^i.$$

DEFINITION 2. The Dyer-Lashof algebra R is the quotient algebra F/J where F is the free associative algebra generated by $\{Q^r, \beta Q^{r+1} | r \geq 0\}$ $[by\ \{Q^r | r \geq 0\}]$, and J is the ideal of F consisting of all elements which annihilate every element of every infinite loop space.

3. **Results on** BU **and** BU. Recall that $H^*(BU) = P\{c_n | n \geq 1\}$ as algebras where c_n is the Chern class of degree $2n$, $c_0 = 1$ and $\psi(c_n) = \sum_{i=0}^n c_i \otimes c_{n-i}$. Let $a_{2n} = (c_1^n)^*$ and $p_{2n} = c_n^*$ in the dual basis of the basis of $H^*(BU)$ which consists of monomials in the Chern classes. Then $H_*(BU) = P\{a_{2n} | n \geq 1\}$ as algebras with $\psi(a_{2n}) = \sum_{i=0}^n a_{2i} \otimes a_{2n-2i}$, and $\{p_{2n} | n \geq 1\}$ is a basis for the primitive elements of $H_*(BU)$. For $r \geq 0$ we have the duals of the Dyer-Lashof operations $Q_*^r : H^*(BU) \rightarrow H^*(BU)$ of degree $-2r(p-1)$ [of degree

$-r$]. Theorems 3 and 4 are our main results on BU.

THEOREM 3. *In* $H_*(BU)$ *for* $r \geq 0$ *and* $n \geq 1$,

$$Q^r(\mathrm{p}_{2n}) = (-1)^{r+n}(n - 1, r - n)\mathrm{p}_{2n+2r(p-1)}$$
$$[Q^{2r}(\mathrm{p}_{2n}) = (n - 1, r - n)\mathrm{p}_{2n+2r}],$$

and

$$Q^r(a_{2n}) = (-1)^{r+n+1}(n, r - n - 1)a_{2n+2r(p-1)} + decomposables$$
$$[Q^{2r}(a_{2n}) = (n, r - n - 1)a_{2n+2r} + decomposables].$$

THEOREM 4. *In* $H^*(BU)$ *for* $r \geq 0$ *and* $n \geq 1$,

$$Q_*^r(c_n) = (-1)^{r+n}(n - r(p - 1) - 1, pr - n)c_{n-r(p-1)}$$

$$[Q_*^{2r}(c_n) = (n - r - 1, 2r - n)c_{n-r}].$$

Theorem 3 is proved by using the comultiplicative Cartan formula, the Nishida relations and Bott periodicity. An algorithm can then be given for computing $Q^r(a_{2n})$ $[Q^{2r}(a_{2n})]$ by induction on $n+r(p-1)$ and for fixed $n+r(p-1)$ by induction on n using the following seven properties that the Q^r satisfy on $H_*(BU)$: (a), (b), (c), (e), (f), (h) of Theorem 1 and Theorem 3. Theorem 4 is now proved by defining homomorphisms R_*^s on $H^*(BU)$ by

$$R_*^s(c_n) = (-1)^{s+n}(n - s(p - 1) - 1, ps - n)c_{n-s(p-1)}$$

$$[R_*^{2s}(c_n) = (n - s - 1, 2s - n)c_{n-s}]$$

and insisting that the R_*^s satisfy the multiplicative Cartan formula. Then the dual maps R^s can be shown to satisfy the above seven properties. Hence $R^s = Q^s$ for all $s \geq 0$ by the algorithm for computing the Dyer-Lashof operations on $H_*(BU)$.

Recall that $H_*(\mathrm{BU}) = H_*(BU) \otimes Z_p(Z)$ as Hopf algebras where $Z_p(Z)$ is the group algebra of the integers over Z_p. Elements of $H_*(\mathrm{BU})$ are written $x \otimes [i]$ for $x \in H_*(BU)$ and $i \in Z \subset Z_p(Z)$. Since the canonical inclusion $BU \to \mathrm{BU}$ induces the map $x \to x \otimes [0]$ in homology for $x \in H_*(BU)$, the following theorem together with our knowledge of the Dyer-Lashof operations on $H_*(BU)$ tell us how the Dyer-Lashof operations act on $H_*(\mathrm{BU})$.

THEOREM 5. *In* $H_*(\mathrm{BU})$ *for* $n \geq 1$, $Q^n(1 \otimes [1]) = \gamma_n(\mathrm{p}_{2(p-1)}) \otimes [p]$ $[Q^{2n}(1 \otimes [1]) = a_{2n} \otimes [2]]$ *where* $\gamma_n(\mathrm{p}_{2(p-1)})$ *is* $(c_{p-1}^n)^*$ *in the dual basis of the basis of* $H^*(BU)$ *which consists of monomials in the Chern classes.*

We will now sketch the applications of the preceding theorems. Define the R-indecomposables of $H_*(BU)$ by

$$Q_R H_*(BU) = IH_*(BU)/[(IR)(IH_*(BU)) + (IH_*(BU))^2].$$

The following theorem is a consequence of Theorem 3.

THEOREM 6. $\{a_{2k} | k \geq 1, \ k = p^e h, \ h = s(p-1)+r, \ 1 \leq r \leq p-1, \ s = \sum_{i=0}^n s_i p^i, \ 0 \leq s_i \leq p-1, \ \bar{s}_i = s_i$ if $s_i \neq 0, \ \bar{s}_i = p$ if $s_i = 0$ and if $s \neq 0$ then $s_n \neq 0$ and $r \geq \bar{s}_0 \geq \cdots \geq \bar{s}_n\}$ $[\{a_{2^k} | k \geq 0\}]$ is a Z_p-basis for $Q_R H_*(BU)$.

If A is a connected Hopf algebra and $f: A \to A$ is a morphism of Hopf algebras then f is said to be locally nilpotent if for all $a \in A$ with deg $a > 0$ there is a positive integer $n(a)$ with $f^{n(a)}(a) = 0$ where $f^0 = 1$, $f^1 = f$ and $f^{k+1} = f \circ f^k$ for $k \geq 1$. In this case every element of $Z_p[[f]]$ is a well-defined Hopf algebra endomorphism of A where addition is given by Whitney sum, i.e., $g + h = \Phi \circ g \otimes h \circ \Psi$. The composite map $\phi: BU \to BSp \to BU$ induces a locally nilpotent endomorphism ϕ_* of $H_*(BU; Z_2)$. $Z_2[[\phi_*]]$ is the ring of all A-Hopf algebra endomorphisms of $H_*(BU; Z_2)$ and the ring of all AR-Hopf algebra endomorphisms of $H_*(BU; Z_2)$. If p is odd then $BU_p = \prod_{i=0}^{p-2} BU_{p,i}$ as infinite loop spaces by J. F. Adams [1, Lecture 4]. There is a locally nilpotent AR-Hopf algebra endomorphism ϕ_i of $H_*(BU_{p,i})$ for $0 \leq i \leq p-2$ such that $Z_p[[\phi_i]]$ is the ring of all A-Hopf algebra endomorphisms of $H_*(BU_{p,i})$ and the ring of all AR-Hopf algebra endomorphisms of $H_*(BU_{p,i})$.

4. Results on O and SO. Recall that $H_*(SO; Z_p) = E\{d_{4n+3} | n \geq 0\}$ as Hopf algebras if p is an odd prime and $H_*(SO; Z_2) = E\{u_n | n \geq 1\}$ as algebras with $u_0 = 1$ and $\psi(u_n) = \sum_{i=0}^n u_i \otimes u_{n-i}$. Hence $H^*(SO; Z_p) = E\{d_{4n+3}^* | n \geq 0\}$ as Hopf algebras if p is an odd prime and $H^*(SO; Z_2) = P\{e_{2n+1} | n \geq 0\}$ as Hopf algebras. We use the convention that the $(n+1)$st Pontryagin class suspends to $-d_{4n+3}^*$. Let $\{p_{2n-1} | n \geq 1\}$ be the primitive elements of $H_*(SO; Z_2)$. From the analogue of Theorem 4 for $H^*(BO)$ we deduce the following three results by using the cohomology suspension map.

THEOREM 7. In $H^*(SO)$ for $r \geq 0$ and $n \geq 0$,

$$Q_*^r(d_{4n+3}^*) = (-1)^r(2n - r(p-1) + 1, pr - 2n - 2)d_{4n-2r(p-1)+3}^*$$

$$[Q_*^{2r}(e_{2n+1}) = (n - r, 2r - n - 1)e_{2n-2r+1}].$$

COROLLARY 8. In $H_*(SO; Z_p)$ for p an odd prime, $r \geq 0$ and $n \geq 0$,

$$Q^r(d_{4n+3}) = (-1)^r(2n + 1, r - 2n - 2)d_{4n+2r(p-1)+3}.$$

COROLLARY 9. *In* $H_*(SO; Z_2)$ *for* $r \geq 0$ *and* $n \geq 1$,

$$Q^r(u_n) = (n, r - n - 1)u_{r+n} + decomposables$$

and

$$Q^{2r}(p_{2n-1}) = (n - 1, r - n)p_{2n+2r-1}.$$

THEOREM 10. *In* $H_*(SO; Z_2)$ *for* $r \geq 0$ *and* $n \geq 1$,

$$Q^r(u_n) = \sum_{a+b+c=r+n} (n - a, r - n - b - 1)u_a u_b u_c.$$

PROOF. This theorem follows from the first assertion of Corollary 9, the comultiplicative Cartan formula and the binomial coefficient identities of J. Adem [2, p. 233].

Recall that $H_*(O; Z_2) = H_*(SO; Z_2) \otimes Z_2(Z_2)$ as Hopf algebras. For $x \in H_*(SO; Z_2)$ write x and $x \otimes [-1]$ to designate elements in the two components of $H_*(O)$. The multiplicative Cartan formula, Theorem 10 and the following theorem give complete information on the Dyer-Lashof operations on $H_*(O; Z_2)$.

THEOREM 11. *In* $H_*(O; Z_2)$, $Q^n(1 \otimes [-1]) = u_n$ *for* $n \geq 0$.

5. **Results on** B **Im** J **and Im** J. Throughout this section p will be an odd prime. J. Stasheff [11] has constructed infinite loop spaces B Im J and Im J. He has shown that $H^*(B$ Im $J) = H^*(BO)_{p,0} \otimes H^*(SU)_{p,0}$ and $H^*(\text{Im } J) = H^*(BO)_{p,0} \otimes H^*(SO)_{p,1/2(p-3)}$ as Hopf algebras with each factor closed under p^r, for all $r \geq 0$. Recall that $H^*(BO)_{p,0} = P\{q_n | n \geq 1\}$ as algebras with deg $q_n = 2n(p-1)$, $q_0 = 1$ and $\psi(q_n) = \sum_{i=0}^n q_i \otimes q_{n-i}$. Then q_n are the Wu classes which are defined by $q_n = \Phi^{-1} \circ P^n \circ \Phi(1)$ where Φ is the Thom isomorphism.

THEOREM 12. *Let* p *be an odd prime. Then*

$$H^*(B \text{ Im } J) = H^*(BO)_{p,0} \otimes H^*(SU)_{p,0}$$

and

$$H^*(\text{Im } J) = H^*(BO)_{p,0} \otimes H^*(SO)_{p,1/2(p-3)}$$

as Hopf algebras with each factor closed under P^r *and* Q^r, *for all* $r \geq 0$. *In particular for* $r \geq 0$ *and* $n \geq 1$,

$$Q_*^r(q_n) = ((n - r)(p - 1) - 1, pr - n(p - 1))q_{n-r}.$$

Detailed proofs and statements of all the above results and many similar ones will appear elsewhere.

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