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THE P-SINGULAR POINT OF THE P-COMPACTIFICATION FOR $\Delta u = Pu$[1]

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ABSTRACT. By means of the P-algebra $M_P(R)$ of bounded energy-finite Tonelli functions on a Riemannian manifold R, we construct the P-compactification R_P^* of R as a quotient space of the Royden compactification. The P-singular point s_P is explicitly characterized in terms of the density P. The dimension of the space $PBE(R)$ of bounded energy-finite P-harmonic functions on R is shown to exceed exactly by one the cardinality of the P-harmonic boundary Δ_P if $s_P \in \Delta_P$. If $s_P \notin \Delta_P$ one can replace the density P by another Q such that dim $QBE(R) =$ dim $PBE(R)$ and a Q-singular point does not exist.

In the study of the equation $\Delta u = Pu$, $P \geqq 0$, on a Riemannian manifold R, it is useful to consider the algebra $M_P(R)$ of bounded energy-finite Tonelli functions. With $M_P(R)$ one associates the P-compactification R_P^* of R on which every $f \in M_P(R)$ has a continuous extension (Nakai-Sario [4]). An interesting phenomenon is the occurrence of the P-singular point $s \in R_P^*$ defined by $f(s) = 0$ for every $f \in M_P(R)$.

In the present note we construct R_P^* as a quotient space of the Royden compactification R^*. Necessary and sufficient for the existence of an s is that $1 \notin M_P(R)$. If an s exists, it is unique. We shall give an explicit characterization of s in terms of P, thus establishing a link with a property considered by Glasner and Katz [2].

We then show that if s lies on the P-harmonic boundary Δ_P, the cardinality of Δ_P exceeds exactly by one the dimension of the space of bounded energy-finite P-harmonic functions on R.

If s does not lie on Δ_P, it is removable in the sense that there exists a density Q on R without a Q-singular point such that dim $QBE(R)$ $=$ dim $PBE(R)$ $=$ the cardinality of Δ_P.

1. On a smooth Riemannian n-manifold R, $n \geqq 2$, consider P-

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harmonic functions, i.e. solutions of the elliptic partial differential equation

$$\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j}\right) = Pu.$$

Here $x=(x^1, \cdots, x^n)$ is a local coordinate, (g^{ij}) the inverse of the matrix (g_{ij}) of the fundamental metric tensor of R, g the determinant of (g_{ij}), and P ($\not\equiv 0$) a nonnegative continuous function on R.

Denote by $M_P(R)$ the algebra of bounded Tonelli functions f on R with finite energy integrals $E_R(f)=E_R(f, f)$. Here the inner product $E_R(f, g)$ is defined by

$$E_R(f, g) = \int_R \left[\sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} + Pfg \right] dV,$$

with dV the volume element $*1$.

Let $f \in M_P(R)$. Given a regular subregion Ω of R, construct the function u on R such that $u \equiv f$ on $R-\Omega$ and $\Delta u = Pu$ on Ω. The energy principle (Royden [5]) reads

$$E_R(u) \leqq E_R(f), \qquad u \in M_P(R).$$

If $g \in M_P(R)$ and $g \equiv 0$ on $R-\Omega$, then $E_R(g, u)=0$.

2. Denote by $M(R)$ the Royden algebra and by R^* the Royden compactification of R (cf. e.g. Chang-Sario [1] and Sario-Nakai [6]). In view of $M_P(R) \subset M(R)$ every function $f \in M_P(R)$ has a continuous extension to R^*.

For $x, y \in R^*$ set $x \sim y$ if $f(x)=f(y)$ for all $f \in M_P(R)$. Clearly "$\sim$" is an equivalence relation. Denote by R_P^* the quotient space $R^*/\sim$. Let $\pi_P: R^* \to R_P^*$ be the natural projection.

PROPOSITION 1. *The space R_P^* endowed with the quotient topology is a compact Hausdorff space and contains R as a connected open dense subset.*

PROPOSITION 2. *Every function in $M_P(R)$ has a continuous extension to R_P^*, and $M_P(R)$ separates points in R_P^*.*

We shall call R_P^* the *P-compactification* and $M_P^*(R)$ the *P-algebra* of R. For the continuations of $f \in M_P(R)$ to R^* and R_P^* we use the same notation f.

P-regularity can be given the following explicit characterization:

3. A point $x \in R_P^*$ will be called *P-regular* or *P-singular* according as there does or does not exist a function $f \in M_P(R)$ with $f(x) \neq 0$. By

virtue of Proposition 2 a P-singular point is unique, if it exists.

THEOREM 1. *A point $x\in R_P^*$ is P-regular if and only if the density function P has a finite integral at x, i.e. there exists an open neighborhood U of x in R_P^* with $\int_{U\cap R} P\,dV<\infty$.*

PROOF. If x is P-regular, there exists a function $f\in M_P(R)$ with $f(x)\neq 0$. Choose $\epsilon>0$ such that $|f(x)|>\epsilon$. Then $U=\{y\in R_P^* \mid |f(y)|>\epsilon\}$ is an open neighborhood of x in R_P^*. Since

$$\int_{U\cap R} P\,dV \leqq \frac{1}{\epsilon^2}\int_{U\cap R} Pf^2\,dV \leqq \frac{1}{\epsilon^2} E_R(f),$$

P has a finite integral at x.

Conversely suppose that there exists an open neighborhood U of x in R_P^* with $\int_{U\cap R} P\,dV<\infty$. Since $R^*-\pi_P^{-1}(U)$ and $\pi_P^{-1}(x)$ are disjoint closed sets in R^*, we can choose a function $g\in M(R)$ such that $0\leqq g\leqq 1$, $g|\pi_P^{-1}(x)\equiv 1$, and $g|R^*-\pi_P^{-1}(U)\equiv 0$. Then we have

$$\begin{aligned}\int_R Pg^2\,dV &= \int_{R\cap\pi_P^{-1}(U)} Pg^2\,dV + \int_{R-\pi_P^{-1}(U)} Pg^2\,dV\\ &\leqq \int_{R\cap\pi_P^{-1}(U)} P\,dV = \int_{R\cap U} P\,dV < \infty.\end{aligned}$$

Thus $g\in M_P(R)$ and $g(x)=1$, i.e. x is P-regular.

A point $s\in R_P^*$ is P-singular if and only if $\int_{U\cap R} P\,dV=\infty$ for each open neighborhood U of s in R_P^*.

REMARK. If there exist no P-singular points, then we have the special case $R_P^*=R^*$ studied in Royden [**5**]. In our note we assume that s exists. The concept of a P-singular point was introduced in Nakai-Sario [**4**], and the term "P has a finite integral at x" in Glasner-Katz [**2**].

4. We write $f=\text{BE-lim}_n f_n$ on R if the sequence $\{f_n\}$ is uniformly bounded on R, converges to f uniformly on compact subsets of R, and $E_R(f_n-f)\to 0$ as $n\to\infty$. In view of the BD-completeness of Royden's algebra $M(R)$ (e.g. Sario-Nakai [**6**]) it is not difficult to see that the P-algebra $M_P(R)$ is BE-complete.

Let $\Delta_P=\pi_P(\Delta)$ and denote by $M_{PO}(R)$ the space of functions in $M_P(R)$ with compact supports in R, and by $M_{P\Delta}(R)$ the space of BE-limits in $M_P(R)$ of functions in $M_{PO}(R)$. As in the case of the potential subalgebra $M_\Delta(R)$ (cf. [**3**]) we have the duality:

PROPOSITION 3. $M_{P\Delta}(R)=\{f\in M_P(R)\,|\,f\equiv 0$ *on* $\Delta_P\}$.

PROOF. It suffices to show that

$$M_{P\Delta}(R) = \{f \in M_P(R) \mid f \equiv 0 \text{ on } \Delta\}.$$

Since $M_{P\Delta}(R) \subset M_\Delta(R)$, $M_{P\Delta}(R) \subset \{f \in M_P(R) \mid f \equiv 0 \text{ on } \Delta\}$ (cf. [3]). Conversely, suppose that $f \in M_P(R)$ vanishes identically on Δ. Since $M_P(R)$ is a lattice, we may assume that $f \geqq 0$. Choose a sequence $\{f_n\}$ of functions in $M(R)$ with compact supports in R such that $0 \leqq f_n \leqq f$ and $f = \text{BD-lim}_n f_n$ on R. By Lebesgue's dominated convergence theorem

$$\int_R Pf^2\, dV = \lim_{n \to \infty} \int_R Pf_n^2\, dV.$$

Consequently $f \in M_{P\Delta}(R)$ as desired.

COROLLARY. *$M_{P\Delta}(R)$ is an ideal of $M_P(R)$.*

5. We turn to the vector space $PBE(R)$ of bounded energy-finite P-harmonic functions on R.

We maintain (for Royden's compactification cf. Glasner-Katz [2]):

THEOREM 2. *The vector space $PBE(R)$ is m-dimensional if and only if the P-harmonic boundary Δ_P consists of $m+1$ points whenever $s \in \Delta_P$. If s does not lie on Δ_P, then* dim $PBE(R)$ *equals the cardinality of Δ_P.*

For the proof we first establish the othogonal decomposition:

LEMMA 1. $M_P(R) = PBE(R) \oplus M_{P\Delta}(R)$.

PROOF. Let $f \in M_P(R)$. Since $M_P(R)$ is a vector lattice we may assume that $f \geqq 0$ on R.

For a regular exhaustion $\{R_n\}$ of R consider the functions $u_n \in M_P(R)$ such that $u_n \in PBE(R_n)$ and $u_n \equiv f$ on $R - R_n$. By the energy principle (cf. 1),

$$E_R(u_n) \leqq E_R(f) < \infty,$$
$$E_R(u_n) = E_R(u_{n+p}) + E_R(u_{n+p} - u_n)$$

for all n, $p \geqq 1$. Hence $\{u_n\}$ is E-Cauchy. Since it is uniformly bounded on R, we may assume that it converges to a P-harmonic function, uniformly on compact subsets of R (cf. Royden [5]).

Set $u = \text{BE-lim}_n u_n$ and $g = \text{BE-lim}_n (f - u_n)$ on R. Then $f = u + g$ is the desired decomposition. Its uniqueness is obvious by the definition of $M_{P\Delta}(R)$.

LEMMA 2. *$R \in O_{PBE} - O_G$ if and only if $\Delta_P = \{s\}$.*

PROOF. If $\Delta_P = \{s\}$, $M_P(R) = M_{P\Delta}(R)$ and $PBE(R) = \{0\}$.

Conversely, suppose that there exists a P-regular point x in Δ_P. Choose open neighborhoods U, V of s in R_P^* such that $x \notin U$ and $\overline{V} \subset U$. Since $\pi_P^{-1}(\overline{V})$ and $\pi_P^{-1}(R_P^* - U)$ are disjoint closed sets in R^* we can construct an $f \in M_P(R)$ with $0 \leq f \leq 1$, $f | \pi_P^{-1}(\overline{V}) \equiv 0$, and $f | \pi_P^{-1}(R_P^* - U) \equiv 1$.

Let $f = u + g$ be the decomposition in Lemma 1. Then u is a nonconstant PBE-function and therefore $R \notin O_{PBE} - O_G$.

PROOF OF THEOREM 2. Let $\{x_1, \cdots, x_m\}$ be a finite subset of $\Delta_P - s$. As in the proof of Lemma 2, we can construct nonconstant functions u_i in $PBE(R)$ with $u_i(x_j) = \delta_{ij}$. Since the u_i are linearly independent, dim $PBE(R) = \infty$ whenever Δ_P is an infinite set.

Suppose that the cardinality of Δ_P is $m+1$ and that $s \in \Delta_P$. For any $u \in PBE(R)$, $u - \sum_{i=1}^m u(x_i) u_i \in PBE(R) \cap M_{P\Delta}(R) = \{0\}$ and we conclude that dim $PBE(R) = m$ is the cardinality of $\Delta_P - s$.

The proof in the case in which the cardinality of Δ_P is finite and $s \notin \Delta_P$ is the same.

6. We have seen that the dimension of the space $PBE(R)$ is equal to the cardinality of the P-harmonic boundary whenever the P-singular point s does not lie on Δ_P. Thus the existence of s in this case is, in a sense, of little significance as far as the relation of $PBE(R)$ and Δ_P is concerned. It is natural to ask: Can one replace the density P by another, Q, such that dim $PBE(R) =$ dim $QBE(R)$, and a Q-singular point does not exist?

First we prove:

THEOREM 3. *The P-singular point s lies on $R_P^* - \Delta_P$ if and only if there exists a PBE-function u on R such that $u \equiv 1$ on Δ_P.*

PROOF. The necessity is trivial since $PBE(R) \subset M_P(R)$. For the sufficiency choose an $f_x \in M_P(R)$ for a given $x \in \Delta_P$ such that $f_x \geq 0$ and $f_x(x) > 0$. Since Δ_P is compact we can construct a function $f \in M_P(R)$ with $f \geq 0$ and $f | \Delta_P > 0$. Set $\alpha = \min_{\Delta_P} f > 0$, and let $\alpha^{-1}(f \cap \alpha) = u + g$ be the decomposition in Lemma 1. Then u has the required property.

THEOREM 4. *If P, Q are densities on R which coincide on an open neighborhood U of Δ in R^*, then* dim $PBE(R) =$ dim $QBE(R)$.

PROOF. First we show that Δ_P and Δ_Q have the same cardinality. Let $\pi_P : R^* \to R_P^*$ be the natural projection and let $\pi_P(x) \neq \pi_P(y)$ for $x, y \in \Delta$. Then there exists a function $f \in M_P(R)$ with $f(x) \neq f(y)$. Choose an open neighborhood V of Δ in R^* such that $\overline{V} \subset U$ and a function $g \in M(R)$ such that $0 \leq g \leq 1$, $g | \bar{V} \equiv 1$, and $g | R^* - U \equiv 0$. Clearly $fg \in M_Q(R)$ and $(fg)(x) \neq (fg)(y)$, i.e. $\pi_Q(x) \neq \pi_Q(y)$. We infer that the cardinalities of Δ_P and Δ_Q coincide, and therefore dim $PBE(R) = \infty$ if and only if dim $QBE(R) = \infty$.

Let the common cardinality of Δ_P and Δ_Q be $k < \infty$. If the P-singular point s_P belongs to Δ_P, choose $x \in \Delta$ such that $\pi_P(x) = s_P$. Then it is easily seen that $\pi_Q(x)$ is the Q-singular point and $\pi_Q(x) \in \Delta_Q$. By Theorem 2 it follows that dim $PBE(R) =$ dim $QBE(R) = k - 1$ (resp. k) if $s_P \in \Delta_P$ (resp. $s_P \notin \Delta_P$).

If a P-singular point s_P exists but does not lie on Δ_P, then it may be called a "removable" P-singular point in the following sense:

THEOREM 5. *If the P-singular point s_P lies on $R_P^* - \Delta_P$, there exists a density Q on R such that* dim $QBE(R) =$ dim $PBE(R)$ *and* $\int_R Q\, dV < \infty$.

PROOF. Choose open neighborhoods U, V of s_P in R_P^* such that $\bar{V} \subset U$ and $\bar{U} \cap \Delta_P = \emptyset$. Since $\pi_P^{-1}(\bar{V})$ and $R^* - \pi_P^{-1}(U)$ are disjoint closed subsets of R^* there exists a function $f \in M_P(R)$ with $0 \leq f \leq 1$, $f | \pi_P^{-1}(\bar{V}) \equiv 0$, and $f | R^* - \pi_P^{-1}(U) \equiv 1$.

Set $Q = f^2 P$. Then $\int_R Q\, dV = \int_R Pf^2\, dV \leq E_R(f) < \infty$, and by Theorem 4 we have dim $QBE(R) =$ dim $PBE(R)$.

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