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CONVERGENCE, SUMMABILITY, AND UNIQUENESS OF MULTIPLE TRIGONOMETRIC SERIES

BY J. MARSHALL ASH[1] AND GRANT V. WELLAND[2]

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1. Relationships between methods of convergences and the growth of coefficients. It was shown by Paul J. Cohen [1] that if a multiple trigonometric series converges regularly at almost every point of the k-torus $T^k = [-\pi, \pi] \times \cdots \times [-\pi, \pi]$, then its coefficients $a_n = a_{n_1,\ldots,n_k}$ cannot exhibit exponential growth. A particular form of regular convergence is square convergence. Consideration of double series of the form

$$\sum_{n=1}^{\infty} \phi(n)(1 - \cos x)^n e^{iny}$$

shows that Cohen's seemingly gross estimates cannot be improved. For by a suitable choice of the $\phi(n)$ the series may be made square convergent almost everywhere while having coefficients which grow faster than any given sequence whose growth is less than exponential.

THEOREM 1. *If a multiple trigonometric series converges unrestrictedly rectangularly on a set, then the coefficients are necessarily bounded; furthermore,* $a_n = a_{n_1,\ldots,n_k} \to 0$ *as* $\min\{|n_1|, \cdots, |n_k|\} = \|n\| \to \infty$.

Again this theorem is best possible. The proof is by induction and makes use of

LEMMA 1. *If a polynomial* $P(e^{ix})$ *of degree* n *is bounded for all* $x \in E \subset [0, 2\pi)$ *by a bound* B, *where* $|E| =$ *Lebesgue measure of* $E = \delta > 0$, *then there is a number* $c = c(\delta, n)$ *such that* $|P(e^{ix})| \leq c$ *for every* x.

The lemma is an easy consequence of the Lagrange interpolation formula and a lemma of Paul Cohen's [1, p. 41]. Another consequence of Lemma 1 is

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THEOREM 2. *If a multiple trigonometric series* Σ *converges unrestrictedly rectangularly on a subset* E *of* T^k, *then there is a set* $F\subset E$, $|F| = |E|$ *such that all the rectangular partial sums of* Σ *are bounded on* F. *(The bound may vary from point to point.) In particular, if* $E=T^k$, *then* $F=E$.

To appreciate that this theorem is trivial only if $k=1$, the reader should consider the numerical double series given by $a_{0n}=n$, $a_{1n}=-n$, $n=1, 2, 3, \cdots$, $a_{mn}=0$ otherwise which converges unrestictedly rectangularly to 0, but has unbounded rectangular partial sums. The proof of Theorem 2 is also an induction which depends heavily on Lemma 1.

2. Relationships between modes of convergence and summability.

THEOREM 3. *If a multiple trigonometric series* Σ *converges unrestrictedly rectangularly on a set* E, *then* Σ *is spherically Abel summable to the same values on* F, *where* E *and* F *are as in Theorem* 2.

This theorem is an easy consequence of Theorem 2 and the fact that Pringsheim convergence (unrestricted rectangular convergence together with bounded rectangular partial sums) implies spherical Abel summability for numerical multiple series. (We say $\sum a_n$ is spherically Abel summable to s if $\sum a_n e^{-|n|h}=A(h)$ exists for every $h>0$ and $\lim_{h\to 0+} A(h)=s$.)

Theorem 3 is somewhat surprising since different modes of convergence are often incompatible. For example, the double series

$$\sum_{p=2}^{\infty} 3p \exp[ip^2x] (\sin y)^{2p} \exp[i(p^2-2p)y]$$

is square convergent almost everywhere but restrictedly rectangularly convergent at no point; while the double series

$$\sum_{p=1}^{\infty} 2^p \exp[i4^px] (\sin y)^{2^p}$$

is restrictedly rectangularly convergent almost everywhere but both circularly and triangularly divergent everywhere. We do have, however,

THEOREM 4. *If a multiple trigonometric series converges on* E, *then it is* $(C, 1, 1)$ *summable on* F, *where* E *and* F *are as in Theorem* 2.

The proof is again Theorem 2 together with a classical theorem concerning numerical series. There are many similar classical theorems in which the hypothesis of bounded partial sums may be dropped. It would be repetitious to list any more of them here.

3. Uniqueness of multiple trigonometric series.

THEOREM 5. *Let the double trigonometric series* Σ *be unrestrictedly rectangularly convergent everywhere on* T^2 *to the finite-valued Lebesgue integrable function* $f(x, y)$. *Then* Σ *is the Fourier series of* f.

The proof of Theorem 5 involves the use of a uniqueness theorem concerning spherical Abel summability due to Victor Shapiro ([5] or [6, pp. 65–78]). The hypotheses in Shapiro's theorem that must be verified are

(i) Σ is spherically Abel summable everywhere to $f(x, y)$,
and

(ii) the sum of the moduli of the coefficients of Σ whose indices lie in an annulus of thickness one and radius R is $o(R)$ as R tends to infinity.

The first hypothesis is satisfied because of Theorem 3 while the second follows from Theorem 1. Condition (ii) does not follow from Theorem 1 if the dimension k is >2 which is why the theorem is stated only in two dimensions. A somewhat more general version of Theorem 5 involving upper and lower limits actually holds but it is too complicated to state here. In higher dimensions, we have the following uniqueness theorem.

THEOREM 6. *Suppose for every* $(x_1, \cdots, x_k) = \mathbf{x}$,

$$\lim_{\|(N_1,\cdots,N_k)\|\to\infty} \sum_{n_1=0}^{N_1} \cdots \sum_{n_{k-1}=0}^{N_{k-1}} \sum_{n_k=-N_k}^{N_k} a_{n_1,\cdots,n_k} \exp[i(n_1x_1 + \cdots + n_kx_k)]$$
$$= \sum a_n \exp[in\cdot\mathbf{x}] = 0.$$

Then all the $a_{n_1,\ldots,n_k}$ *are zero.*

The statement is given in a fairly specific fashion since it is not clear at the present time whether the restriction that all but one of the sums be one-sided is necessary. A uniqueness theorem for multiple trigonometric series of power series type is an immediate corollary.

To prove Theorem 6 form a sequence of functions $L_0(\mathbf{x}) = \sum a_n \exp[in\cdot\mathbf{x}]$, $L_1(\mathbf{x}), \cdots, L_k(\mathbf{x})$ where each L_i is obtained by two formal integrations in the ith coordinate. The facts that the second ith partial symmetric derivative of L_i is equal to L_{i-1} and that L_i is a continuous function of its ith variable are used to establish that the continuous function $L_k(\mathbf{x})$ has a special form from whence it follows that L_k and hence L_0 have all coefficients equal to zero.

Another approach to the theory of uniqueness may be made by considering $L_k(x)$ directly and working with a kth symmetric derivative directly. This method, which has been pursued by Geiringer [2] and Žak [8] does yield some interesting partial results but not as yet a theory of uniqueness.

A fourth approach to uniqueness is via generalized "integrals" wherein the first "integral" is a vector (Shapiro [7]), the second "integral" is a matrix, the third a three tensor, and so forth. A partial result arising from this study is that

$$\lim_{h\to 0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left(\frac{m \sin mh + n \sin nh}{(m^2 + n^2)h} \right)^{\alpha} = s$$

whenever $\alpha > 2$ and $\sum\sum a_{mn} = s$ in the sense of Pringsheim.

4. On the almost everywhere summability of double Fourier series. The Fourier series $S[f]$ of $f(x, y) \in L(T^2)$ is said to be summable (C, α, β) at (x, y) if

$$\lim_{\|(m,n)\|\to\infty} \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x - s, y - t) K_m^{\alpha}(s) K_n^{\beta}(t) ds dt = f(x, y),$$

where $K_m^{\alpha}(s)$ is the αth Fejér kernel. (See Zygmund [9, p. 94].)

Jessen, Marcinkiewicz, and Zygmund [4] have proved that if $\alpha > 0$ and $\beta > 0$, and if $f \in L \log^+ L$ on T^2, then f is summable (C, α, β) at almost every point of T^2. If $\beta = 0$, we set

$$K_n^0(t) = D_n(t) = \frac{\sin(n + \frac{1}{2})t}{2 \sin t/2}.$$

THEOREM 7. *If $f \in L(\log^+ L)^2$ on T^2 and if $\alpha > 0$, then f is summable $(C, \alpha, 0)$ almost everywhere on T^2.*

The proof uses maximal functions and is an easy corollary of a one-dimensional maximal theorem of Hunt [3] concerning convergence in the theory of functions of one variable.

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DePaul University, Chicago, Illinois 60614