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HOMOTOPY THEORY OF RINGS AND ALGEBRAIC K-THEORY

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ABSTRACT. Algebraic K-theory is interpreted in terms of standard homotopy notions applied to the category of rings. Representability of the functors K^{-i} is discussed.

The object of this announcement is to indicate how the algebraic K-theory of [1] and [2] can be explained as homotopy theory in a precise sense. Some terminology of homotopy theory was used in both these articles, but the analogy turns out to be very far reaching. We work in the category $\mathfrak{B}$ of Banach rings complete in their quasinorm [2]; morphisms are bounded homomorphisms. The terminology of [2] will be assumed. From $\mathfrak{B}$ one constructs the category Hot-$\mathfrak{B}$ whose objects are those of $\mathfrak{B}$ and morphisms are homotopy classes of bounded maps (some care must be observed in defining Hot-$\mathfrak{B}$ since homotopy is per se neither transitive nor symmetric but does behave well with respect to compositions). Denote Hot-$\mathfrak{B}(A, B)$ by $[A, B]$.

DEFINITION 1. If X and $A \xrightarrow{f} B$ are in $\mathfrak{B}$, one says that f is an X-fibration if for each $n \geqq 1$, $E^n f$ induces a surjection $\mathfrak{B}(X, E^n A) \to \mathfrak{B}(X, E^n B)$.

LEMMA 1. *For all X in $\mathfrak{B}$, $A\{x\{ \to A$ and $EA \to A$ given by "$x \to 1$" are X-fibrations.*

DEFINITION 2.[1] The mapping cone $C(g)$ of $g: B \to C$ is the fibre product in the diagram

$$\begin{array}{ccc} C(g) & \longrightarrow & EB \\ \downarrow g_1 & & \downarrow \\ A & \xrightarrow{g} & B \end{array}$$

LEMMA 2. *For any X there is an exact sequence of pointed sets*

$$[X, C(g)] \to [X, A] \to [X, B].$$

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[1] I believe a more appropriate terminology for EA and ΩA would have been the cone and suspension of A respectively.

One may iterate the construction $C(g)$ to get the diagram

$$\cdots \to C(g_n) \xrightarrow{g_{n+1}} C(g_{n-1}) \to \cdots \to C(g) \xrightarrow{g_1} A \xrightarrow{g} B$$

and the corresponding exact Puppe sequence of homotopy sets

$$\to [X, C(g_n)] \to [X, C(g_{n-1})] \to \cdots \to [X, A] \to [X, B].$$

Assume now that X is a cogroup in $\mathcal{B}$. That is, there is a morphism $X \to X \amalg X$ in $\mathcal{B}$ such that $(1_X \amalg 0) \cdot \Delta = 1_X$, $(0 \amalg 1_X) \circ \Delta = 1_X$ and $(\Delta \amalg 1_X) \circ \Delta = (1_X \amalg \Delta) \circ \Delta$. Then $\mathcal{B}(X, A)$ is a group for all A.

PROPOSITION 1. *If X is a cogroup in $\mathcal{B}$, then the functor $\overline{X} = \mathcal{B}(X, \cdot)$: $\mathcal{B} \to$ Groups is a Mayer-Vietoris Functor in the sense of* [1]. *In addition $[X, A] = \kappa_1^{\bar{X}}(A)$, where in the terminology of* [1] *(using the appropriate path ring $EA = xA\{x\{)\kappa_1^{\overline{X}}(A)$ is defined by the exact sequence*

$$\overline{X}(EA) \to \overline{X}(A) \to \kappa_1^{\bar{X}}(A) \to 1.$$

THEOREM 1. *Assume again that X is a cogroup in $\mathcal{B}$ and the diagram*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence in $\mathcal{B}$ with g an X-fibration. Then the exact Puppe sequence above is precisely the exact K-theory sequence of [1]

$$\cdots \to \kappa_{n+1}^{\bar{X}}(A) \to \kappa_{n+1}^{\bar{X}}(B) \to \kappa_{n+1}^{\bar{X}}(C) \to \kappa_n^{\bar{X}}(A) \to \cdots.$$

(Again, EA must be suitably interpreted in [1] for nondiscrete rings.)

We may apply these notions to the functors Gl_n and Gl.

PROPOSITION 2. Gl_n *is representable by* gl_n *in* $\mathcal{B}$. Gl *is pro-representable in $\mathcal{B}$ by* gl. *Both* gl_n *and* gl *are cogroups, the latter in pro-$\mathcal{B}$.*

COROLLARY. *For any A in $\mathcal{B}$ we have canonical isomorphisms $\kappa_1^{\mathrm{Gl}_n}(A) \cong [\mathrm{gl}_n, A]$ and $K^{-1}(A) = \kappa({}^{1}_{\mathrm{Gl}}A) = [\mathrm{gl}, A]$, where the last equation is interpreted in the pro-homotopy category. Furthermore, the exact Puppe sequence for $X = \mathrm{gl}$ in Theorem 1 is precisely the exact sequence of* [2] *of the functors K^{-n}.*

We can also consider the representability of the functors K^{-n} of [2].

LEMMA 3. *The loop ring functor $A \to \Omega A$ has an adjoint Σ in pro-$\mathcal{B}$. One has pro-$\mathcal{B}$ $(\Sigma A, B) \cong \mathcal{B}(A, \Omega B)$ and $[\Sigma A, B] = [A, \Omega B]$.*

As a word of caution it should be noted that Σ is not the suspension functor S of [2]. From Lemma 3 one deduces

PROPOSITION 3. *The functor* K^{-n} $(n>0):\mathcal{B}\to Ab$ *is pro-represented by* $\Sigma^{n-1}\mathrm{gl}$.

We show in addition that κ_i^G is pro-representable in $\mathcal{B}$, where G is a Mayer-Vietoris functor which is an algebraic group. (The definition of κ_i^G in [1] is modified as in Proposition 1 above in the nondiscrete case.)

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