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KOEBE SETS FOR UNIVALENT FUNCTIONS WITH TWO PREASSIGNED VALUES

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1. **Introduction.** Let $\mathfrak{M}_M$ denote the set of all functions $f(z)$ that are analytic and univalent in the unit disc Δ and satisfy the conditions $f(0)=0$, $f(z_0)=z_0$, and $|f(z)|\leqq M$, where z_0 is a fixed point of Δ, $z_0\neq 0$, and where M is fixed, $1<M\leqq\infty$.

Although the class $\mathfrak{M}_\infty$ has been a popular one to study, very little seems to have been done with $\mathfrak{M}_M$. We aim to correct this oversight by beginning a study of $\mathfrak{M}_M$. In this paper we obtain the exact value of the "Koebe constant" for $\mathfrak{M}_M$ and we determine the Koebe sets for

(i) the set $\mathfrak{M}_M^*$ consisting of those elements $f(z)$ of $\mathfrak{M}_M$ for which $f(\Delta)$ is starlike with respect to the origin, and

(ii) the set $\mathfrak{M}_\infty^\alpha$ consisting of those members $f(z)$ of $\mathfrak{M}_\infty$ for which $f(\Delta)$ is convex in the direction $e^{i\alpha}$.

2. **Main results.** By the *Koebe constant* for $\mathfrak{M}_M$ we mean the radius of the largest disc, center at the origin, that lies in the set $\bigcap[f(\Delta)\,|\,f\in\mathfrak{M}_M]$, the Koebe set for $\mathfrak{M}_M$.

THEOREM 1. *The Koebe constant for $\mathfrak{M}_M$ is given by*

$$r(\mathfrak{M}_M) = 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2},$$
$$\delta = \frac{M - |z_0|}{1 - |z_0|}. \tag{1}$$

This result is sharp.

PROOF. First, there is no loss of generality here if z_0 is taken to be real and positive. Hence we set $z_0 = r_0 > 0$. Now we obtain the domain Ω^* from the domain $\Omega\equiv f(\Delta)$ by a circular symmetrization with respect to the half-line $[0, r_0, \infty)$. The domain Ω^* contains the origin

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and it contains the point r_0; moreover, it is contained in a domain D_h which is the disc $[w \mid |w| < M]$ slit along the segment $(-M, -h]$. Well-known monotonic properties of the hyperbolic distance give us the inequalities

$$\begin{aligned} \text{arc tgh } r_0 \equiv \rho(0, r_0, \Omega) &\geqq \rho(0, r_0, \Omega^*) \\ &\geqq \rho(0, r_0, D_h) \equiv \text{arc tgh } |\phi(r_0)|, \end{aligned} \tag{2}$$

where $w = \phi(z)$ is a function that maps Δ conformally onto D_h subject to the condition $\phi(0) = 0$. A computation involving $\phi(z)$ shows that (2) holds if (1) holds, with $z_0 = r_0$. The equality sign in (1) holds for the function $f(z)$ defined by

$$\frac{f(z)}{[M - e^{-i\alpha}f(z)]^2} = \left(\frac{1 - |z_0|}{M - |z_0|}\right)^2 \frac{z}{(1 - e^{-i\alpha}z)^2}, \qquad z_0 = r_0 e^{i\alpha}.$$

This completes the proof of Theorem 1.

REMARK. For $M = \infty$, (1) gives us a result due to Lewandowski [3], and for $z_0 = 0$, (1) yields a result due to Pick [5].

Now we shall determine the Koebe set for $\mathfrak{M}_M^*$, the set of elements of $\mathfrak{M}_M$ that map Δ onto domains that are starlike with respect to the origin.

First we recall some facts about Koebe sets. If $\mathcal{E}$ is a nonempty set of functions $f(z)$, analytic in Δ, then the *Koebe set* of $\mathcal{E}$ is the set $\mathcal{K}(\mathcal{E}) \equiv \bigcap [f(\Delta) \mid f \in \mathcal{E}]$, [1]. However, for the set $\mathfrak{M}_M^*$, Krzyż and Złotkiewicz found another characterization of $\mathcal{K}(\mathfrak{M}_M^*)$. Let Δ_M denote the disc $[w \mid |w| < M]$ and let $\mathcal{G}$ denote the set of all subdomains D of Δ_M that (i) are starshaped with respect to the origin, and (ii) contain the fixed point z_0. For $D \in \mathcal{G}$, let $g(w, z_0, D)$ be the Green's function with pole at z_0 and let $\mu(w_0)$ be defined by

$$\mu(w_0) = \text{lub}[g(0, z_0, D) \mid D \in \mathcal{G}, w_0 \in \Delta_M \backslash D]. \tag{3}$$

Then the result due to Krzyż and Złotkiewicz, alluded to above, is that

$$\mathcal{K}(\mathfrak{M}_M^*) \equiv \left[w \,\middle|\, \mu(w) < \log \frac{1}{|z_0|} \right] \tag{4}$$

holds [2].

Now if we make use of (3) and (4), then we obtain the following result.

THEOREM 2. *The set $\mathcal{K}(\mathfrak{M}_M^*)$ is determined by the condition*

$$|w - w_0| \left| \frac{M^2 - w\bar{z}_0}{M + |w|} \right| + \frac{1}{|w|} \left| \frac{Mw + z_0|w|}{M + |w|} \right| < \frac{1}{2}(1 + |z_0|^2),$$

and the Koebe set $\mathcal{K}(\mathfrak{M}_\infty^*)$ *is determined by the inequality*

$$|w - z_0| + |w| < \tfrac{1}{2}(1 + |z_0|^2). \tag{5}$$

REMARK. The elliptic domain (5) is a well-known one due to Rogosinski [6].

The formula in (4) can be applied to other subclasses of $\mathfrak{M}_M$. For example, we have found the following result.

THEOREM 3. *The Koebe set* $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ *is determined by the inequality*

$$1+[A(1+\cos 2\theta)+(B^2-1)\cos 2\theta+BC\sin 2\theta]^{1/2}<\frac{1-(1-D^2)^{1/2}}{|z_0|^2},$$

where

$$h = |w|, \qquad d = |w - z_0|, \qquad \theta = \alpha - \arg z_0,$$

$$A = \frac{2hd}{(h+d)^2}, \qquad B = \frac{h-d}{|z_0|}, \qquad D = \left|\frac{z_0}{h+d}\right|,$$

$$C = [(1 - D^2)(2A + D^2 - 1)]^{1/2}.$$

The set $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ *is a simply-connected Jordan domain if* $|z_0|\,[1+|\sin\theta|\,]^{1/2}<1, \theta\neq 0, \pi$, *while* $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ *is the union of two disjoint simply-connected Jordan domains, which are symmetric with respect to the point* $\frac{1}{2}z_0$, *if* $1<|z_0|\,[1+|\sin\theta|\,]^{1/2}, \theta\neq 0, \pi$.

REMARK. For $z_0 = 0$ we obtain

$$\mathcal{K}(\mathfrak{M}_\infty^{\pi/2}) \equiv [w \mid \delta \mid w \mid (\,|w| + |\operatorname{Im} w|\,) < 1],$$

which is related to a result due to McGregor [4].

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