EQUATIONAL AXIOMS FOR CLASSES OF LATTICES

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1. Equational axiom problems. This note presents a general method for solving a number of problems in the equational theory of lattices. Current interest in this theory stems from Jónsson's important discovery [7], building on earlier work of Birkhoff, that a lattice sharing the algebraic identities common to a given class K of lattices is more tightly bound to K than would be expected for most other kinds of algebraic systems. In this context, a natural question for any such class K is the following "axiom problem."

A.P.(K): *Find a set of equational axioms for K, i.e., a set Σ of identities, common to the members of K, of which all other such identities are lattice-theoretic consequences.*

An equivalent requirement on Σ is that the class of lattices defined by Σ coincide with K^e, the smallest equational class (class definable by identities) containing K.

The problem A.P.(K) is to be viewed as a practical one; the solution is to be constructed explicitly starting from some given definition of K. McKenzie [9], for example, has given just such an explicit solution of A.P.$(\{L\})$ for each finite lattice L.

The general method to be developed below solves the axiom problems of all classes of lattices in the following list, among many others.

(a) PP, the class of all projective planes (viewed as lattices of flats). More generally,

(b) $PP(\mathcal{E})$, the class of projective planes subject to a given list $\mathcal{E}$ of excluded configurations. An example is the class of Desarguesian planes, for which solutions to the axiom problem have been given by Schützenberger [10] and Jónsson [6, Theorem 7.1].

(c) $Lth(m)$, the class of all lattices of length at most m, i.e., lattices in which the longest chain has at most $m+1$ elements. This axiom problem was posed by Jónsson, who later solved the case $m=2$ [8].

(d) $Wth(m)$, the class of all lattices of width at most m, i.e., lattices

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in which every totally unordered subset has at most m elements. This axiom problem was also posed by Jónsson; the case $m = 2$ was solved by McKenzie [10].

(e) $Br(m)$, the class of all lattices of breadth at most m, i.e., lattices such that every join of $m+1$ elements is redundant.

(f) Pl, the class of "planar" lattices, i.e., lattices which can be order-embedded in a product of two chains.

(g) $K_1 \cup K_2$, the union of two equational classes of lattices, each defined by a given finite set of identities (McKenzie).

Additionally, Wille [12] raises the topic of equational axioms for the classes $M \cap Lth(m)$ and $M \cap Wth(m)$ of *modular* lattices of length or width at most m. (Here M denotes the class of all modular lattices.) Jónsson [8] in effect solves the case A.P.$(M \cap Wth(3))$. The promised solution of A.P.(K) for *any* of the nonmodular classes K of (c)–(g) above will yield a solution to A.P.$(M \cap K)$ when augmented by the modular identity.

For each of the axiom problems previously solved, the axioms given have been finite in number. McKenzie [9] and the author [1] have, however, given examples of classes K for which A.P.(K) has *no* finite solution. In fact, it can be shown that, in at least these specific cases, no finite solution exists: $Lth(m)$ for $m \geq 4$; $Wth(m)$ for $m \geq 4$; $M \cap Wth(m)$ for $m \geq 5$; $Br(m)$ and $M \cap Br(m)$ for $m \geq 2$; Pl; and $K_1 \cup K_2$ for certain equational classes K_1, K_2 each defined by a single identity. On the other hand, PP and $M \cap Lth(m)$, for all m, do give finite solutions. (Details will appear elsewhere.) The general method of §§2–3 below for solving axiom problems always yields an infinite list of identities, reducible (not necessarily constructively) to a finite list when a finite solution exists.

The methods of this paper can be generalized to solve axiom problems relative to any equational class of algebras to which Jónsson's results of [7] apply. Especially promising are applications to orthomodular lattices, Heyting algebras, and lattice-ordered groups.

A general reference for lattice theory is [3]; for universal algebra, [5].

2. **A general problem.** A common method of solution to all the axiom problems listed in §1 is to be found at the end of this twisting trail of reasoning:

(1) The most powerful tool relevant to such problems is Jónsson's main theorem of [7].

(2) The classes of lattices to which Jónsson's theorem applies most naturally are those which are closed under the formation of sub-

lattices, homomorphic images, and ultraproducts. Following the *ad hoc* terminology of [1], let us call such classes "stable."

(3) It is a result in model theory that a stable class is definable by sentences of a simple form, namely by universally quantified disjunctions of polynomial equations (UDE's).

(4) Happily, most of the classes of lattices on our list are stable, and it is possible to give explicit defining UDE's. The other classes can be expanded to stable ones without changing their axiom problems.

(5) Therefore, all the listed axiom problems are but instances of the axiom problem for a much more general class, namely, the problem A.P.(**Lat** Δ), where Δ is any given set of UDE's and **Lat** Δ denotes the class of all lattices satisfying all the sentences in Δ (lattice models of Δ).

(6) This very general axiom problem can immediately be reduced to the case where Δ consists of a single UDE δ. A solution to this reduced problem will be constructed in the next section.

Amplification for (1), (3), (4), (6). For (1). A statement of Jónsson's result of [7, §3], adapted for our purposes, is as follows. (For terminology, see [3] or [5].)

THEOREM 2.1 (JÓNSSON). *Let S be the class of all subdirectly irreducible lattices. If K is stable, then $S \cap K^e = S \cap K$; in fact, K^e is the unique equational class E such that $S \cap E = S \cap K$.*

For (3). A class closed under formation of ultraproducts, sublattices, and isomorphic images is axiomatic [5, Corollary 1, p. 258]; an axiomatic class closed under formation of sublattices and homomorphic images is definable by positive universal sentences [5, Corollary 2, p. 275], each of which is trivially equivalent to a conjunction of UDE's. Conversely, if Δ is a set of UDE's, **Lat** Δ is stable. Any identity, of course, is trivially a UDE.

For (4). The key method is to apply the following principle from [1]. If P is any finite partly ordered set and $N(P)$ is the class of all lattices which contain no order-isomorphic copy of P, then $N(P)$ is stable. For example, $Lth(m) = N(C_{m+2})$, where C_{m+2} is an $(m+2)$-element chain; $Wth(m) = N(U_{m+1})$, where U_{m+1} is a totally unordered set of $m+1$ elements; $Br(m) = N(B_{m+1})$, where B_k is the partly ordered set consisting of 1-element and $(k-1)$-element subsets of a k-element set; $Pl = \bigcap_i N(Q_i)$, where Q_1, Q_2, $\cdots$ correspond to "comparability cycles" [2]. For any given P, $N(P)$ can be defined by an explicit UDE as follows.

LEMMA 2.2. *Let $p(1)$, $\cdots$, $p(n)$ be the elements of the finite partly or-*

dered set P. For each i, let $g_i(x)$ denote the lattice polynomial $g_i(x_1, \cdots, x_n)$
$= \bigvee_j x_j$ with j ranging over $\{j : p(j) \leqq p(i)\}$. Then $N(P) = \text{Lat}\{\delta\}$, where
δ is the UDE *$(\forall x_1, \cdots, x_n) OR_{i,k}[g_i(x) = g_i(x) \bigvee g_k(x)]$ with (i, k)*
ranging over $\{(i, k) : p(k) \not\leqq P(i)\}$.

Here *OR* denotes logical disjunction, to avoid confusion with lattice operations. The proof of the lemma is easy and will be omitted.

For a treatment of the nonstable classes *PP* and *PP*(ε) in terms of *N*( ), see [1]. The only remaining class on our list is $K_1 \cup K_2$; this class is definable by conjunctions of pairs of identities, each of which can be turned into an equivalent UDE by using disjoint sets of variables and taking quantifiers outside.

For (6). If Δ is a set of UDE's, a solution to A.P.(Lat Δ) is obtained merely by lumping together all the identities involved in the solutions Σ_δ of the various problems A.P.(Lat δ) for δ∈Δ. This follows from the computation

$$\text{Lat}(\bigcup_{\delta \in \Delta} \Sigma_\delta) = \bigcap_{\delta \in \Delta} \text{Lat } \Sigma_\delta = \bigcap_{\delta \in \Delta} (\text{Lat } \delta)^e = (\bigcap_{\delta \in \Delta} \text{Lat } \delta)^e = (\text{Lat } \Delta)^e,$$

in which the third equality is an application of Theorem 2.1, and **Lat δ** means **Lat**$\{\delta\}$.

3. A solution to the general problem. As shown in the last section, it suffices to solve A.P.(Lat δ), where δ is a single UDE. Such a δ can be written as

$$\delta = (\forall x_1, \cdots, x_m)[f_1(x) = g_1(x) \text{ OR } \cdots \text{ OR } f_n(x) = g_n(x)].$$

Replacing f_i, g_i by $f_i \wedge g_i$, $f_i \vee g_i$ if necessary, we may assume that $f_i(x) \leqq g_i(x)$ for all *i*.

The polynomials appearing in the solution of A.P.(Lat δ) will be constructed as a three-layered composition, with the inner layer consisting of the f_i and g_i. The middle layer will be built from the polynomials p_k defined for $k = 0, 1, 2, \cdots$ by

$$p_k(t; z_1, \cdots, z_k) = (\cdots (((t \vee z_1) \wedge z_2) \vee z_3) \cdots,$$

ending in $\cdots) \wedge z_k$ for *k* even, $\cdots) \vee z_k$ for *k* odd.

The outer layer will consist simply of $p_{2n}(t; z)$ for *n* as in δ. It will be apparent from the proof of Theorem 3.1 that this type of construction is indeed a natural one.

For δ as given, then, let us construct for each $k = 0, 1, 2, \cdots$ an identity σ_k, as follows. Let *Z* be an $n \times k$ array of variables z_j^i, let r_i stand for the expression $p_k(f_i(x); z_1^i, \cdots, z_k^i)$, and let s_i stand for $p_k(g_i(x); z_1^i, \cdots, z_k^i)$. Let σ_k be defined by

$$\sigma_k = (\forall u, v, x, Z)[p_{2n}(u; r_1, s_1, \cdots, r_n, s_n) = p_{2n}(v; r_1, s_1, \cdots, r_n, s_n)].$$

Let Σ_δ be the collection $\{\sigma_k : k = 0, 1, 2, \cdots\}$.

THEOREM 3.1. Σ_δ *solves the axiom problem for the class* **Lat** δ.

The proof, given below, will depend on three properties of the polynomials p_k. For a lattice L let $\mathrm{Quo}(L)$ denote the set of nontrivial "formal quotients" of L, i.e., pairs $(a, b) \in L \times L$ with $b > a$, written instead as b/a. For b/a, $d/c \in \mathrm{Quo}(L)$, write $b/a \rightarrow d/c$ if there exist $k \geq 0$ and elements $w_1, \cdots, w_k \in L$ such that $c = p_k(a; w_1, \cdots, w_k)$, $d = p_k(b; w_1, \cdots, w_k)$. This is shown without difficulty to be equivalent to the assertion that d/c is weakly projective into b/a [**4**, Definition 2.1].

LEMMA 3.2. *Let L be a lattice.*

(i) *If $c, a_1, \cdots, a_k \in L$ and $a_i = a_{i+1}$ for some i, then the value $p_k(c; a_1, \cdots, a_k)$ does not depend on the choice of c.*

(ii) *If $d/c \in \mathrm{Quo}(L)$ and $u = c$ or $u = d$, then $p_k(u; c, d, c, d, \cdots, c, d)$* $= u$.

(iii) $\rightarrow$ *is a quasi-ordering on $\mathrm{Quo}(L)$ and is directed if L is subdirectly irreducible.*

PROOF. (i) and (ii) are immediate, as are reflexivity and transitivity in (iii). Directedness is verified as follows. For $b/a \in \mathrm{Quo}(L)$, let $\theta(a, b)$ denote the congruence relation generated by identifying a and b. [**4**, Lemma 2.1] states that if $\theta(c, d) \subseteq \theta(a, b)$, then there exists a chain $c = c_0 < c_1 < \cdots < c_n = d$ with $b/a \rightarrow c_i/c_{i-1}$ for all $i > 0$. Further, if L is subdirectly irreducible, there exists a *smallest* congruence relation $\theta(c, d)$ [**3**, Corollary 2, p. 140]. Thus, if b/a, b'/a' are any two given elements of $\mathrm{Quo}(L)$, there are corresponding chains $\{c_i\}$, $\{c'_j\}$ from c to d. Choose (i, j) to be a minimal pair of indices such that $c_i \wedge c'_j > c$. Then for $e = c_i \wedge c'_j$, $b/a \rightarrow c_i/c_{i-1} \rightarrow e/c$, b'/a' $\rightarrow c'_j/c'_{j-1} \rightarrow e/c$, so that e/c is an upper bound for b/a, b'/a' in $\mathrm{Quo}(L)$.

PROOF OF THEOREM 3.1. We must show that $(\mathbf{Lat}\ \delta)^e = \mathbf{Lat}\ \Sigma_\delta$. By Theorem 2.1, it is sufficient to verify that for a subdirectly irreducible lattice L, $L \in \mathbf{Lat}\ \delta$ if and only if $L \in \mathbf{Lat}\ \Sigma_\delta$.

Suppose $L \in \mathbf{Lat}\ \delta$. Then for any k and for any substitution of elements for the variables $\{x_i\}$, $\{z^i_j\}$, we have the following situation: For some i, the evaluated f_i and g_i are equal. Then the evaluated r_i and s_i are also equal, and hence by Lemma 3.2-(i), the value of $p_{2n}(c; r_1, s_1, \cdots, r_n, s_n)$ does not depend on the choice of $c \in L$— exactly what σ_k asserts! Thus $L \in \mathbf{Lat}\ \Sigma_\delta$.

Suppose conversely that $L \notin \mathbf{Lat}\ \delta$. Then, interpreting variables as

elements of L, we know that there exist $x_1, \cdots, x_m \in L$ such that $f_i(x) < g_i(x)$ for all i. Write $a_i = f_i(x)$, $b_i = g_i(x)$. By Lemma 3.2-(iii), there exists $d/c \in \mathrm{Quo}(L)$ such that $b_i/a_i \rightarrow d/c$ for all i. Then, for a large enough k, there exist suitable $z_j^i \in L$ such that for all i, $r_i = p_k(a_i; z_1^i, \cdots, z_k^i) = c$, $s_i = p_k(b_i; z_1^i, \cdots, z_k^i) = d$. Setting $u = c$, $v = d$ and applying Lemma 3.2-(ii), we obtain $p_{2n}(u; r_1, s_1, \cdots, r_n, s_n) = c$, $p_{2n}(v; r_1, s_1, \cdots, r_n, s_n) = d$, a violation of σ_k. Therefore $L \in$ **Lat Σ_δ**. The proof is thus complete.

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