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DEFORMATIONS OF LIE SUBGROUPS

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1. **Introduction.** This note is an announcement of results concerning the local deformation theory of subgroups of a Lie group. Let G be a real (resp. complex) Lie group and let M be a real (resp. complex)-analytic manifold. Roughly speaking, an analytic family of Lie subgroups of G, parametrized by M, is an analytic submanifold $\mathcal{H}$ of $G\times M$ such that each "fibre" H_t $(t\in M)$ is a Lie subgroup of G; here the "fibre" H_t is defined by $\mathcal{H}\cap(G\times\{t\})=H_t\times\{t\}$. (See §2 for a precise definition of an analytic family of Lie subgroups.) Our main result concerning such families is

THEOREM A. *Let $\mathcal{H}=(H_t)_{t\in M}$ be an analytic family of Lie subgroups of G, let $t_0\in M$ and let $H=H_{t_0}$. Let K be a Lie subgroup of H such that the component group K/K^0 is finitely generated and such that the Lie group cohomology space $H^1(K, \mathfrak{g}/\mathfrak{h})$ vanishes. Then there exists an open neighborhood U of t_0 in M and an analytic map $\beta: U\to G$ such that $K\subset\beta(t)H_t\beta(t)^{-1}$ for every $t\in U$.*

Here $\mathfrak{g}$ (resp. $\mathfrak{h}$) denotes the Lie algebra of G (resp. H) and the K-module structure of $\mathfrak{g}/\mathfrak{h}$ is determined by the adjoint representation of K on $\mathfrak{g}$.

Theorem A generalizes the result of A. Weil [6, p. 152] which states that if Γ is a discrete, finitely generated subgroup of G such that $H^1(\Gamma, \mathfrak{g})=0$, then Γ is "rigid". It also generalizes results of the author [4], [5] on deformations of subalgebras of Lie algebras to the case of Lie subgroups. The proof of Theorem A depends heavily on the analyticity assumptions, although we suspect that the C^∞ version of the theorem is also true.

If G acts as an analytic transformation group on the analytic manifold M and if all orbits of G on M have the same dimension, then it can be shown that the connected isotropy groups $(G_t^0)_{t\in M}$ form an analytic family of Lie subgroups of G, and hence Theorem A applies. For example, let K be a maximal compact subgroup of G_t^0. Then

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$H^1(K, \mathfrak{g}/\mathfrak{g}_t) = 0$ and thus there exists a neighborhood U of t such that G_s^0 contains a subgroup conjugate to K for every $s \in U$. For the case of algebraic transformation groups (over C) one gets considerably stronger theorems along the same line.

2. **Analytic families of Lie subgroups.** Analytic manifolds and Lie groups are taken over either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. Analytic submanifolds and Lie subgroups are defined as in [2]; in particular analytic submanifolds and Lie subgroups are not required to have the topology induced by the ambient manifold. The Lie algebra of a Lie group G will be denoted by the corresponding German letter $\mathfrak{g}$ and the connected component of the identity in G will be denoted by G^0.

DEFINITION 2.1. Let G be a Lie group and let M be an analytic manifold. Then an *analytic family of Lie subgroups of* G, parametrized by M, is an analytic submanifold $\mathcal{H}$ of $G \times M$ which satisfies the following conditions:

(a) Let $\pi_M: \mathcal{H} \to M$ denote the composition $\mathrm{pr}_M \circ i$, where $i: \mathcal{H} \to G \times M$ is the inclusion map and pr_M is the projection $G \times M \to M$. Then π_M is surjective and is a submersion.

(b) Each fibre $\pi_M^{-1}(t)$ $(t \in M)$ is of the form $H_t \times \{t\}$, where H_t is a Lie subgroup of G.

(c) Let $\mathcal{H} \times_M \mathcal{H} = \{(a, b) \in \mathcal{H} \times \mathcal{H} \mid \pi_M(a) = \pi_M(b)\}$ and let $m: \mathcal{H} \times_M \mathcal{H} \to \mathcal{H}$ and $s: \mathcal{H} \to \mathcal{H}$ be defined by $m((x, t), (y, t)) = (xy, t)$ and $s(x, t) = (x^{-1}, t)$. Then m and s are analytic maps.

It follows from the definition that the function $t \to \dim H_t$ is constant on each component of M.

3. **Sketch of the proof of Theorem A.** Let F denote either $\mathbf{R}$ or $\mathbf{C}$. Since the problem is local, we may assume that M is an open neighborhood of 0 in F^r and that $t_0 = 0$. We let W be a vector subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$. If $\pi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then the restriction π_W of π to W is a vector space isomorphism of W and $\mathfrak{g}/\mathfrak{h}$; we define a K-module structure on W by transferring the K-module structure on $\mathfrak{g}/\mathfrak{h}$ to W by means of π_W. Let $\eta: K \to \mathrm{GL}(W)$ denote the corresponding representation.

The following lemma is proved by a straightforward application of the implicit function theorem.

LEMMA 3.1. *There exists an open neighborhood U of $H \times \{0\}$ in $H \times M$ and an analytic map $\psi: U \mapsto W$ such that the following conditions hold for all $(x, t) \in U$;*

(a) $\psi(x, 0) = 0$;

(b) $(\exp \psi(x, t))x \in H_t$;

(c) *the map* $(x, t) \mapsto ((\exp \psi(x, t)x, t)$ *is an analytic diffeomorphism of* U *onto an open neighborhood of* $H \times \{0\}$ *in* $\mathcal{H}$.

The function ψ is called the *normal displacement function* of the family $\mathcal{H}$. The germ of ψ in a neighborhood of $H \times \{0\}$ is uniquely determined by the family $\mathcal{H}$.

For each $x \in K$, the map $\psi_x : t \mapsto \psi(x, t)$ is an analytic map of an open neighborhood U_x of 0 in M into W. Thus we may expand ψ_x in a convergent power series about 0,

$$\psi_x(t) = \sum_{m=1}^{\infty} P_m(x, t),$$

where, for each m, $t \mapsto P_m(x, t)$ is the restriction to U_x of a homogeneous polynomial map of degree m of F^r into W; denote this homogeneous polynomial map by $Q_m(x)$. If $\mathcal{P}_m$ denotes the vector space of all homogeneous polynomial maps of F^r into W, then $Q_m : K \to \mathcal{P}_m$ is an analytic map. Let s denote the smallest integer j such that $Q_j \neq 0$. Q_s is called the *first nonvanishing infinitesimal displacement along* K *of the analytic family* $\mathcal{H}$.

We define a K-module structure on $\mathcal{P}_m$ as follows: If $x \in K$ and $Q \in \mathcal{P}_m$, then $x \cdot Q = \eta(x) \circ Q$. It follows easily from the hypothesis that $H^1(K, \mathcal{P}_m) = 0$.

PROPOSITION 3.2. *Q_s is a one cocycle of K.*

Since $H(K, \mathcal{P}_s) = 0$, it follows from Proposition 2.2 that there exists $\phi_s \in \mathcal{P}_s$ such that $P_s(x, t) = \phi_s(t) - x \cdot \phi_s(t)$ for $x \in K$ and $t \in U_x$. Using this, it can be shown that if we replace the analytic family $\mathcal{H}$ by the family

$$\mathcal{H}' = (\exp \phi_s(t)) H_t (\exp - \phi_s(t))_{t \in M},$$

then the first nonvanishing infinitesimal displacement along K of the analytic family $\mathcal{H}'$ is of degree $\geqq s+1$.

Continuing inductively, we can define an infinite family $(\phi_u)_{u=s,\, s+1,\, \ldots}$, where ϕ_u is a homogeneous polynomial map of F^r into W of degree u, such that the following condition holds: let $u \geqq s$, let $\phi^u = \phi_s + \phi_{s+1} + \cdots + \phi_u$ and let $\mathcal{H}_u$ denote the analytic family $(\exp \phi^u(t)) H_t (\exp - \phi^u(t))_{t \in M}$; then the first nonvanishing infinitesimal displacement of the family $\mathcal{H}_u$ is of degree greater than u.

Let ϕ denote the formal power series map of M into W given by $\phi = \phi_s + \phi_{s+1} + \cdots$. If ϕ converges in a neighborhood of 0 and if $\beta(t) = \exp \phi(t)$ then it is easy to see that β satisfies the conditions of Theorem A. At this point, we need to use a recent theorem of M.

Artin [1]. Very roughly, Artin's theorem says that if a finite number of analytic equations admit a formal power series solution, then they admit a convergent power series solution. With some work, we can show that Artin's theorem implies that the formal power series ϕ above can be chosen to be convergent, which proves Theorem A.

4. **Applications to analytic transformation groups.** Let the Lie group G act as an anlytic transformation group on the analytic manifold M. If $t\in M$, then the subgroup $G_t=\{g\in G \mid g\cdot t=t\}$ is called the *isotropy group* of G at t; the identity component G_t^0 is the connected isotropy group at t.

PROPOSITION 4.1. *Let G act on M as above and assume that all orbits of G on M have the same dimension. Then the family of connected isotropy groups $(G_t^0)_{t\in M}$ is an analytic family of Lie subgroups of G.*

Thus we see that Theorem A applies to the situation above.

A Lie group G is *reductive* if the component group G/G^0 is finite, if G admits a faithful finite-dimensional analytic representation and if every finite-dimensional analytic representation of G is completely reducible. If G is reductive and if $\rho:G\to \mathrm{GL}(V)$ is an analytic representation of G, then it is easy to see that $H^1(G, V)=0$.

Now let G and M be as in Proposition 4.1, let $t\in M$ and let K be a reductive subgroup of G_t^0. Then Theorem A implies that there exists a neighborhood U of t on M such that G_y^0 contains a conjugate of K for every $y\in U$.

5. **Applications to algebraic transformation groups.** Let G be a complex linear algebraic group and let G act as an algebraic transformation group on the complex algebraic variety M.

PROPOSITION 5.1. *There exists a nonempty, Zariski-open subset U of M such that the family $(G_t)_{t\in U}$ is an analytic family of Lie subgroups of G.*

If S is a complex linear algebraic group, then it is known (see [3]) that S admits a semidirect decomposition $S=R\cdot U$, where U is the unipotent radical of S and R is a reductive algebraic subgroup of S; R is determined to within conjugacy by elements of U. Such a decomposition is called a *Levi decomposition* of S.

Now let (G, M) be an algebraic transformation space as above and, for every $t\in M$, let $G_t=R_t\cdot U_t$ be a Levi decomposition of G_t. Then the following theorem is a consequence of Theorem A and Proposition 5.1.

THEOREM B. *There exists a finite family* $X_1, \cdots, X_n$ *of Zariski-locally closed subsets of* M *such that the following conditions hold:*

(a) $M=\bigcup_{j=1}^{n} X_j$.

(b) *For each* j, X_j *is a Zariski-open subset of* $M-(\bigcup_{i=1}^{j-1} X_i)$.

(c) *If* $x, y\in X_j$, *then* R_x *and* R_y *are conjugate.*

(d) *For each* j, *the family* $(U_t)_{t\in X_j}$ *is an analytic family of Lie subgroups of* G.

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