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A SHORT PROOF OF A THEOREM OF PLANS ON THE HOMOLOGY OF THE BRANCHED CYCLIC COVERINGS OF A KNOT

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Let $K \subset S^3$ be a (tame) knot, with complement $C = S^3 - K$, and let $\tilde{C}$ be the infinite cyclic covering of K, i.e. the covering of C corresponding to the commutator subgroup of $\pi_1(C)$. The group of covering translations of $\tilde{C}$ is $H_1(C)$, which is infinite cyclic by Alexander duality; this gives an action of $\boldsymbol{Z}$ on $H_1(\tilde{C})$, and so $H_1(\tilde{C})$ becomes a Λ-module, where Λ is the integral group ring of $\boldsymbol{Z}$. We identify Λ with the ring of polynomials in a single variable t, (positive and negative powers of t being allowed), with integral coefficients. (See [4].)

The k-fold branched cyclic covering of K, M_k $(k \geqq 1)$ is defined by taking the covering of C corresponding to the kernel of the composition:

$$\pi_1(C) \to H_1(C) \cong \boldsymbol{Z} \to \boldsymbol{Z}_k,$$

and branching about K. (For more details, see [1], [4].) M_k is a closed, orientable 3-manifold: for example, M_1 is just S^3.

If $M(t) = (m_{ij}(t))$, $m_{ij}(t) \in \Lambda$, is a presentation matrix for $H_1(\tilde{C})$ as a Λ-module, then it can be shown that a presentation matrix for $H_1(M_k)$ $(k > 1)$ as an abelian group is obtained by substituting for each entry $m_{ij}(t)$, which is some finite formal sum, $\sum_\nu \alpha_\nu t^\nu$, say, the $k \times k$ block $\sum_\nu \alpha_\nu T_k^\nu$, where the summation indicates ordinary matrix addition, and T_k is the $k \times k$ matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 & 1 \cdots 0 \\ \vdots & & & \ddots \vdots \\ 0 & 0 & 0 & 0 \cdots 1 \\ 1 & 0 & 0 & 0 \cdots 0 \end{pmatrix},$$

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(and T_k^0 is defined to be the $k\times k$ identity matrix). (See [2], [4].) Call the matrix obtained from $M(t)$ in this way $M(T_k)$.

Now a geometrical description of $\tilde{C}$, in terms of an orientable surface spanning K, shows that we may take $M(t)$ to be of the form $tV-V^T$, where V is a $2h\times 2h$ matrix over $\mathbf{Z}$ ($h\geqq$ the genus of K) and V^T is the transpose of V. (See [1], [6].) $M(T_k)$ is then $2hk\times 2hk$, but Seifert showed (see [1], [6]) that it is in fact equivalent (in the sense of presenting the same abelian group) to a $2h\times 2h$ matrix, F_k say. In [5], Plans shows that this F_k can be expressed in terms of two matrices P_k and Q_k, which in turn are defined by certain recurrence relations analogous to those defining the Fibonacci numbers. These facts are used to effect a diagonalisation of F_k, from which some interesting general conclusions about $H_1(M_k)$ are drawn, perhaps the most striking being the following:

THEOREM (PLANS). *If k is odd, then $H_1(M_k)$ is a direct double, i.e. $H_1(M_k)\cong G\oplus G$, for some G.*

The proof given in [5] is rather long and involved, and it is the purpose of this note to show that, although F_k is smaller than $M(T_k)$, the above result actually follows very easily from an examination of the big matrix.

PROOF. $M(t)=tV-V^T$. So by a suitable sequence of row interchanges and column interchanges, $M(T_k)$ can be brought into the form:

$$\begin{pmatrix} -V^T & V & 0 & 0 & \cdots & 0 & 0 \\ 0 & -V^T & V & 0 & \cdots & 0 & 0 \\ 0 & 0 & -V^T & V & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -V^T & V \\ V & 0 & 0 & 0 & \cdots & 0 & -V^T \end{pmatrix},$$

a $k\times k$ matrix of $2h\times 2h$ blocks.

Now if k is odd, $k=2r+1$ say, a further sequence of row interchanges gives a matrix in which the rows of blocks occur in the order (numbering them according to their positions in the old matrix): $r+1, r+2, \cdots, 2r+1, 1, 2, \cdots, r$. It is easy to see that this new matrix is skew-symmetric. We illustrate the case $k=7$:

$$\begin{pmatrix} 0 & 0 & 0 & -V^T & V & 0 & 0 \\ 0 & 0 & 0 & 0 & -V^T & V & 0 \\ 0 & 0 & 0 & 0 & 0 & -V^T & V \\ V & 0 & 0 & 0 & 0 & 0 & -V^T \\ -V^T & V & 0 & 0 & 0 & 0 & 0 \\ 0 & -V^T & V & 0 & 0 & 0 & 0 \\ 0 & 0 & -V^T & V & 0 & 0 & 0 \end{pmatrix}.$$

But it is well known (see for example [3, p. 52]) that any skew-symmetric $2n \times 2n$ matrix over $\boldsymbol{Z}$ is equivalent to a block diagonal matrix of the form:

$$\sum_{i=1}^{n} \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \qquad (a_i \geqq 0),$$

and hence presents a direct double.

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