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AN ALGEBRA OF GENERALIZED FUNCTIONS ON AN OPEN INTERVAL; TWO-SIDED OPERATIONAL CALCULUS

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Let Ω be an open subinterval of the real line; suppose that $0 \in \Omega$. The purpose of this announcement is to describe an injection of $L^{\text{loc}}(\Omega)$ into a commutative algebra of operators. The injection is a useful substitute for the two-sided Laplace transformation; in case Ω is the whole real line, the injection can be extended to a space $\mathfrak{B}$ of distributions whose supports may be all of $(-\infty, \infty)$ (there are no growth restrictions: see §7). If the distributional derivative of an arbitrary distribution R belongs to the space $\mathfrak{B}$, then R also belongs to $\mathfrak{B}$ and R has an initial value (because R equals a continuous function in some left-neighborhood of the origin). Thus, it is possible to assign initial conditions to the unknown distribution in a differential equation whose right-hand side belongs to the space $\mathfrak{B}$: see 7.3.

1. **Preliminaries.** If $f_1(\ )$ and $f_2(\ )$ belong to the space $L^{\text{loc}}(\Omega)$ (of all the complex-valued functions which are integrable on each compact subinterval of the open set Ω), we denote by $f_1 \wedge f_2(\ )$ the function defined by

$$(1.1)\qquad f_1 \wedge f_2(t) = -\int_t^0 f_1(t-u) f_2(u)\,du \qquad \text{(for all } t \text{ in } \Omega).$$

2. **The space of test-functions.** Let $W\Omega$ be the space of all the complex-valued functions which are infinitely differentiable on Ω and whose every derivative vanishes at the origin. Thus, $w(\ )$ belongs to $W\Omega$ if $w(\ ) \in C^\infty(\Omega)$ and $w^{(k)}(0) = 0$ for every integer $k \geqq 0$.

2.1. *The space of generalized functions.* Let $\mathfrak{A}\Omega$ be the space of all the linear operators A which map $W\Omega$ into $W\Omega$ such that the equation

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$$A(w_1 \wedge w_2)(t) = (Aw_1) \wedge w_2(t) \qquad \text{(for all } t \text{ in } \Omega)$$

holds whenever $w_1(\ )$ and $w_2(\ )$ belong to $W\Omega$. As usual, if $w(\ )$ belongs to $W\Omega$, then $Aw(\ )$ denotes the image of $w(\ )$ under the operator A.

2.2. *The injection.* If $f(\ )$ belongs to $L^{\text{loc}}(\Omega)$ we denote by $\{f(t)\}$ the operator $w(\ ) \mapsto (f \wedge w)'(\ )$ which assigns to each $w(\ )$ in $W\Omega$ the derivative of the function $f \wedge w(\ )$.

THEOREM. *The linear transformation $f(\ ) \mapsto \{f(t)\}$ is one-to-one and maps $L^{\text{loc}}(\Omega)$ into the linear space $\mathfrak{A}\Omega$.*

2.3. *The operational calculus.* The linear space $\mathfrak{A}\Omega$ is a commutative algebra with unit element 1 (the identity operator), multiplication being the usual composition of operators.

If $f(t) = 1$ for $t \in \Omega$, then $\{f(t)\}$ is the identity operator 1 (defined by $1w(\ ) = w(\ )$ for all $w(\ )$ in $W\Omega$). The differentiation operator D (defined by $Dw(\ ) = w'(\ )$ for all $w(\ )$ in $W\Omega$) is an invertible element of the algebra $\mathfrak{A}\Omega$ such that the equation

$$\{f_1 \wedge f_2(t)\} = \{f_1(t)\} D^{-1} \{f_2(t)\} \tag{2.4}$$

holds for every $f_1(\ )$ and $f_2(\ )$ in $L^{\text{loc}}(\Omega)$. The preceding properties imply all the usual operational formulas. For example, if $f(\ )$ is locally absolutely continuous on Ω, then

$$\{f'(t)\} = D\{f(t)\} - f(0)D, \tag{2.5}$$

whence

$$\{\sin t\} = \frac{D}{D^2 + 1} \quad \text{and} \quad \{\cos t\} = D\{\sin t\}. \tag{2.6}$$

We can now solve problems such as

$$y''(t) + y(t) = \sec(\pi t / 2\omega) \qquad (-\omega < t < \omega); \tag{1}$$

to that effect, set $\Omega = (-\omega, \omega)$, $c_0 = y(0)$, $c_1 = y'(0)$, and inject both sides of (1) into $\mathfrak{A}\Omega$; from (2.5) it follows that

$$\{y(t)\} = c_0 D \frac{D}{D^2 + 1} + c_1 \frac{D}{D^2 + 1} + \frac{D}{D^2 + 1} D^{-1} \left\{ \sec \frac{\pi t}{2\omega} \right\};$$

from (2.4), (2.6), and (1.1) it therefore results that

$$y(t) = c_0 \cos t + c_1 \sin t - \int_t^0 [\sin(t - u)] \sec \frac{\pi u}{2\omega} \, du.$$

2.7. *Other operational calculi.* Mikusiński's injection (of $L^{\text{loc}}(0, \infty)$

into the Mikusiński field) is an extension of the Laplace transformation; analogously, our injection $f(\ )\mapsto\{f(t)\}$ can be compared with the two-sided Laplace transformation. However, the two-sided Laplace transforms

$$\mathfrak{L}\{e^{-t}-e^{t}\} \quad\text{and}\quad \mathfrak{L}\{e^{-|t|}\}$$

are restrictions of **the same function** to different vertical strips; contrastingly,

$$\{e^{-t}-e^{t}\} = \frac{2D}{1-D^2} \quad\text{and}\quad \{e^{-|t|}\} = \frac{D^2-D\Sigma}{D^2-1},$$

where $\Sigma=\{\operatorname{sgn} t\}=\{t/|t|\}$. More generally, if $-\infty<\lambda<\infty$ then

$$\{e^{\lambda|t|}\} = \frac{D^2+\lambda D\Sigma}{D^2-1}\,.$$

A problem which is not Laplace transformable is discussed in 7.8.

3. **Direct sum decomposition.** Let $1_+(\ )$ be the Heaviside step function; we set $1_+=\{1_+(t)\}$, $1_-=1-1_+$, and

$$(B\mathfrak{A}) = \{BA : A\in\mathfrak{A}\Omega\} \qquad (\text{when } B\in\mathfrak{A}\Omega).$$

Both $(1_-\mathfrak{A})$ and $(1_+\mathfrak{A})$ are ideals in the algebra $\mathfrak{A}\Omega$, and their direct sum equals $\mathfrak{A}\Omega$:

$$\mathfrak{A}\Omega = (1_-\mathfrak{A})\oplus(1_+\mathfrak{A}).$$

In fact, 1_- (respectively, 1_+) is a projector of $\mathfrak{A}\Omega$ into $(1_-\mathfrak{A})$ (respectively, $(1_+\mathfrak{A})$), $(1_-)^2=1=(1_+)^2$, and $(1_-)(1_+)=0$.

3.1. REMARK. If $h(\ )\in L^{\text{loc}}(\Omega)$ then $\{h_-(t)\}\in(1_-\mathfrak{A})$ and $\{h_+(t)\}\in(1_+\mathfrak{A})$, where $h_+(\ )=1_+(\ )h(\ )$ and

$$h_-(\ ) = h(\ ) - h_+(\ ).$$

4. **Translation properties.** If $-\infty<x<\infty$ we define the function $1^x(\ )$ by $1^x(t)=0$ for $-|x|\leqq t<|x|$, and by $1^x(t)=1$ for all other values of t. Further, we set $1^x=\{1^x(t)\}$.

4.1. THEOREM. *Suppose that* $\alpha>0$ *and* $\lambda\geqq 0$; *if* $h(\ )\in L^{\text{loc}}(\Omega)$ *then the equation*

$$\frac{1^\lambda\{h(t)\}}{1-c1^\alpha} = \left\{\sum_{k=0}^{\infty} c^k[h_-(t+k\alpha+\lambda)+h_+(t-k\alpha-\lambda)]\right\} \tag{4.2}$$

holds for any complex number c.

4.3. *Comments.* The series inside the right-hand bracket is a locally finite sum. Theorem 4.1 is the two-sided extension of Theorem 5.29 in [4]. If $h(\ )$ is a periodic function having period $\alpha > 0$, then

$$\{h(t)\} = \frac{\{[1 - 1^{\alpha}(t)]h(t)\}}{1 - 1^{\alpha}}.$$

5. **The topology.** Let the function-space $W\Omega$ be equipped with the topology of pointwise convergence on the interval Ω; since $\mathfrak{A}\Omega$ consists of mappings of $W\Omega$ into the topological space $W\Omega$, we can equip $\mathfrak{A}\Omega$ with the product topology. The following results have been proved by Harris Shultz: the topology of the linear space $\mathfrak{A}\Omega$ is sequentially complete, locally convex, and Hausdorff; moreover, the multiplication of the algebra $\mathfrak{A}\Omega$ is sequentially continuous.

5.1. *The translation operator.* If $-\infty < x < \infty$ we set $\mathsf{T}_x = \{\mathsf{T}_x(t)\}$, where

$$\mathsf{T}_x(\ ) = -1_-(x)1_-^x(\ ) + 1_+(x)1_+^x(\ ).$$

It turns out that $D\mathsf{T}_x = \lim\{\epsilon^{-1}F_\epsilon(t)\}$ (as $\epsilon \to 0+$), where $F_\epsilon(\ )$ is the characteristic function of the interval $(x, x+\epsilon)$; this indicates that $D\mathsf{T}_x$ corresponds to the Dirac distribution δ_x concentrated at the point x. If $h(\ ) \in L^{\mathrm{loc}}(\Omega)$ is a periodic function of period $\alpha > 0$, then the equation

$$\{h(t)\} \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha} = \left\{h(t) \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t)\right\} \tag{5.2}$$

holds for any complex-valued sequence c_k ($k = 0, \pm 1, \pm 2, \pm 3, \cdots$).

6. **Initial values.** Given A and B in $\mathfrak{A}\Omega$, we shall say that A **equals** B **on an interval** if $Aw(t) = Bw(t)$ for all t in the interval and for all $w(\ )$ in $W(-\infty, \infty)$. For example, any element of $(1_+\mathfrak{A})$ equals 0 on $(-\infty, 0)$.

6.1. Definition. A number c is called an **initial value** (of B) if $c = \lim f(t)$ (as $t \to 0-$) for some function $f(\ )$ such that $\{f(t)\}$ equals B on some interval $(\lambda, 0)$.

6.2. Remarks. If the set of initial values (of B) is not void, it contains a unique element $B(0-)$; we shall see in 7.3 that the operator $DB - B(0-)D$ corresponds to the distributional derivative.

7. **A new space of distributions.** Let $\mathfrak{B}_L$ (respectively, $\mathfrak{B}_+$) be the space of all the Schwartz distributions whose supports are contained in the half-open interval $(-\infty, 0]$ (respectively, $[0, \infty)$). Let $\mathfrak{B}_-$ be

the space of all the elements of $\mathfrak{B}_L$ that are regular in some open neighborhood of the origin; we set

$$\mathfrak{B} = \mathfrak{B}_- + \mathfrak{B}_+.$$

Thus, $\mathfrak{B}$ is the family of all the sums $S+R$, where R is a distribution with Supp $R \subset [0, \infty)$, and where $S \in \mathfrak{B}_-$ (that is, Supp S is contained in $(-\infty, 0]$ and there exists a distribution $\partial^0 f$ corresponding to a function $f(\ )$ such that $S-\partial^0 f$ is zero in some open interval containing the origin). It turns out that $\mathfrak{B}$ is the direct sum $\mathfrak{B}_- \oplus \mathfrak{B}_+$: if $F \in \mathfrak{B}$ there exists a unique pair (F_-, F_+) in $\mathfrak{B}_- \times \mathfrak{B}_+$ such that $F = F_- + F_+$. If $F \in \mathfrak{B}$ and $G \in \mathfrak{B}$ we set

$$F \otimes G = - F_- \star G_- + F_+ \star G_+,$$

where $\star$ is convolution in the sense of [3, p. 384]. By adjoining the multiplication $(F, G) \mapsto F \otimes G$ to the linear space $\mathfrak{B}$ we obtain a commutative ring. Denoting by $\partial^0 f$ the distribution corresponding to a function $f(\ )$, we have $(\partial^0 f_1) \otimes (\partial^0 f_2) = \partial^0 (f_1 \wedge f_2)$.

If $F \in \mathfrak{B}$ and $w(\ ) \in W(-\infty, \infty)$ then $F \otimes (\partial^0 w')$ is the distribution corresponding to a function in $W(-\infty, \infty)$ that we shall denote by $\{F\} w(\ )$; it turns out that $\{F\} w(\ )$ belongs to $W(-\infty, \infty)$. Let $\{F\}$ denote the mapping $w(\ ) \to \{F\} w(\ )$ (of $W(-\infty, \infty)$ into itself); if δ_x is the Dirac distribution, then

$$\{\delta_x\} = D\mathsf{T}_x \qquad (-\infty < x < \infty). \tag{7.1}$$

7.2. THEOREM. *The transformation* $F \mapsto \{F\}$ *is a linear injection of* $\mathfrak{B}$ *into* $\mathfrak{A}(-\infty, \infty)$ *such that* $\{\partial^0 f\} = \{f(t)\}$ *for* $f(\ ) \in L^{\mathrm{loc}}(-\infty, \infty)$ *and*

$$\{F_1 \otimes F_2\} = \{F_1\} D^{-1} \{F_2\} \qquad (\textit{for } F_1 \textit{ and } F_2 \textit{ in } \mathfrak{B}).$$

7.3. THEOREM. *If R is a distribution whose distributional derivative belongs to $\mathfrak{B}$, then the set of initial values (of $\{R\}$: see 6.1) contains a unique element which we shall denote by $\langle R, 0-\rangle$; further,*

$$\{\partial R\} = D\{R\} - \langle R, 0-\rangle D.$$

as usual, ∂R denotes the distributional derivative of R.

7.4. *Differential equations.* Given S in $\mathfrak{B}$, and let a_k $(k=0, 1, 2, \cdots, n)$ be a set of at least two complex numbers; if y is a distribution such that

$$(a_n \partial^n + \cdots + a_1 \partial + a_0) y = S, \tag{7.5}$$

then the distributional derivative $\partial^k y$ belongs to $\mathfrak{B}$ for $0 \leqq k \leqq n$; from

7.3 it therefore follows that we can take into account the initial values $\langle \partial_y^\nu, 0- \rangle$ when $0 \leqq \nu < n$. The equation (7.5) implies that

$$(7.6) \quad \{\partial^k y\} = D^k\{y\} - \sum_{\nu=0}^{k-1} \langle \partial^\nu y, 0- \rangle D^{k-\nu} \qquad (\text{for } 0 \leqq k \leqq n).$$

7.7. THEOREM. *If c_k $(k=0, 1, 2, \cdots, n-1)$ is a sequence of complex numbers, there exists one and only one distribution y satisfying* (7.5) *and the initial conditions*

$$\langle y, 0- \rangle = c_0, \qquad \langle \partial y, 0- \rangle = c_1, \cdots, \qquad \langle \partial^{n-1} y, 0- \rangle = c_{n-1}.$$

7.8. *An example.* The distributional equation

$$(1) \qquad \partial^2 y + y = \sum_{k=-\infty}^{\infty} \delta_{2k\pi}$$

(discussed on p. 128 of [1]) cannot be solved by the method of fundamental (or "elementary") solutions—nor can it be solved by using the finite Fourier transform [5, pp. 335–342]. However, it can readily be solved by injecting it into $\mathfrak{A}(-\infty, \infty)$: from (1), (7.6), and (7.1) it follows that

$$(2) \qquad (D^2 + 1)\{y\} = c_0 D^2 + c_1 D + \sum_{k=-\infty}^{\infty} D\mathrm{T}_{2k\pi},$$

where $c_0 = \langle y, 0- \rangle$ and $c_1 = \langle \partial y, 0- \rangle$. A particular solution results by setting $c_0 = c_1 = 0$; solving (2) for $\{y\}$, we can use (2.6) to obtain

$$\{y\} = \frac{1}{D^2 + 1} D \sum_{k=-\infty}^{\infty} \mathrm{T}_{2k\pi} = \{\sin t\} \sum_{k=-\infty}^{\infty} \mathrm{T}_{2k\pi};$$

from (5.2) it now follows the answer

$$(3) \qquad y(t) = \sin t \sum_{k=-\infty}^{\infty} \mathrm{T}_{2k\pi}(t) = \left(1 + \left[\frac{t}{2\pi}\right]\right) \sin t$$

$(-\infty < t < \infty)$; as usual, $[t/2\pi]$ is the greatest integer $< t/2\pi$. The answer (3) cannot be obtained by Fourier transform methods.

7.9. ACKNOWLEDGMENTS. At the origin of Theorem 7.7 is an article by César de Freitas [2]; his "opérateurs de Heaviside" are linear combinations of functions with distributions of finite order whose supports are locally finite; these distributions form a proper subspace of our space $\mathfrak{B}$. Harris Shultz gave me the idea that $F \otimes G \in \mathfrak{B}$ whenever both $F \in \mathfrak{B}$ and $G \in \mathfrak{B}$.

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