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NORMAL SOLVABILITY FOR NONLINEAR MAPPINGS INTO BANACH SPACES

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Let X be a topological space, Y a Banach space, f a mapping of X into Y. The mapping f is said to be *normally solvable* (following a sort of terminology due to Hausdorff for linear operators) if its image $f(X)$ is closed in Y, with Y given its strong topology. The objective of the theory of normally solvable mappings is to establish conclusions on the fine structure of the image set $f(X)$ from the hypothesis that $f(X)$ is closed in Y together with hypotheses concerning the asymptotic direction set $D_x(f)$ of f at various points x of f, (conclusions which are also described as extensions of the Fredholm alternative to such nonlinear mappings f). The concept of asymptotic direction set is defined as follows:

DEFINITION 1. *Let X be a topological space, Y a Banach space, f a mapping of X into Y, x a given point of X. Then the asymptotic direction set $D_x(f)$ of f at x is the subset of Y defined by*

$$D_x(f) = \bigcap_{\epsilon>0} \mathrm{cl}(\{y \mid y \in Y, y = \xi(f(u) - f(x)),$$
$$\xi \geqq 0, u \in X, \|f(u) - f(x)\| < \epsilon\}),$$

where cl *denotes the closure in the strong topology on Y.*

Under sharper hypotheses, we have the following description of the asymptotic direction set:

PROPOSITION 1. *Let X be a locally convex topological vector space, Y*

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a Banach space, f a mapping of X into Y which is once Gateaux differentiable from X to Y at a given point x of X with differential df_x which is a continuous linear mapping from X to Y. Let $(df_x)^*$ be the dual mapping from Y^* to X^*, $N(df_x^*)$ its nullspace, and $(N(df_x^*))^\perp$ its annihilator in Y. Then:

$$D_x(f) \supset \mathrm{cl}(df_x(X)) = (N(df_x^*))^\perp.$$

Our basic result is the following:

THEOREM 1. *Let X be a topological space, Y a Banach space, f a mapping of X into Y such that $f(X)$ is closed in Y. Let y be a given point in Y, and for $r>0$, let $B_r(y)$ be the closed ball of radius r about y in Y. Suppose that there exists $r>0$ and $p<1$ such that $f^{-1}(B_r(y))$ is nonempty, while for each x in $f^{-1}(B_r(y))$,*

$$\mathrm{dist}(y - f(x), D_x(f)) \leqq p\|y - f(x)\|.$$

Then: y lies in $f(X)$.

A global analogue of Theorem 1 is the following:

THEOREM 2. *Let X be a topological space, Y a Banach space, f a map of X into Y such that $f(X)$ is closed in Y. Suppose that for each y in Y, there exists $r(y)>0$ and $p(y)<1$ such that $f^{-1}(B_{r(y)}(y)) \neq \emptyset$ for all y in Y, while for each x in $f^{-1}(B_{r(y)}(y))$,*

$$\mathrm{dist}(y - f(x), D_x(f)) \leqq p(y)\|y - f(x)\|.$$

Then: $Y = f(X)$.

Using Proposition 1, we obtain the following specializations of these results:

COROLLARY 1 TO THEOREM 1. *Let X be a locally convex topological vector space, Y a Banach space, f a once Gateaux differentiable mapping of X into Y with $f(X)$ closed in Y. Let y be a given element of Y and suppose for an $r>0$ such that $f^{-1}(B_r(y)) \neq \emptyset$ and for a given $p<1$ that for all x in $f^{-1}(B_r(y))$, we have*

$$\|y - f(x) + N(df_x^*)^\perp\|_{Y/N(df_x^*)^\perp} \leqq p\|y - f(x)\|_Y.$$

Then: y lies in $f(X)$.

COROLLARY 1 TO THEOREM 2. *Let X be a locally convex topological vector space, Y a Banach space, f a once Gateaux differentiable mapping of X into Y with $f(X)$ closed in Y. Suppose that the hypotheses of the Corollary 1 to Theorem 1 hold for each y in Y. Then $f(X)$ is the whole of Y.*

Specializing still further by taking $p=0$ and $p(y)\equiv 0$, respectively, we obtain the following:

COROLLARY 2 TO THEOREM 1. *Let X be a locally convex topological vector space, Y a Banach space, f a once Gateaux differentiable mapping of X into Y with $f(X)$ closed in Y. Let y be an element of Y, suppose that $f^{-1}(B_r(y))\neq\emptyset$ for a given $r>0$. Suppose that for each x in $f^{-1}(B_r(y))$ and each y^* in $N(df_x^*)$, we have*

$$(y^*, y - f(x)) = 0.$$

Then: y lies in $f(X)$.

COROLLARY 2 TO THEOREM 2. *Let X be a locally convex topological vector space, Y a Banach space, f a once Gateaux differentiable mapping of X into Y such that $f(X)$ is closed in Y. Suppose that for each x in X, $N(df_x^*)=\{0\}$. Then: $f(X)$ is the whole of Y.*

The special case of Corollary 2 to Theorem 1 in which Y is reflexive, $f(X)$ is assumed to be weakly closed in Y, and $r=\text{dist}(y, f(X))$ was given by Pohozhayev in [6]. The special case of Corollary 2 to Theorem 2 in which Y is uniformly convex was given by Pohozhayev [7]. The result of Theorem 2 for $p(y)=0$ for all y, (which is roughly equivalent to assuming $D_x(f)=Y$ for all x in X), was given by the writer for general Banach spaces Y in Browder [3]. This was extended in Browder [4] to mappings into infinite dimensional manifolds Y with the condition on $D_x(f)$ imposed upon x in $X-N$ only, with the exceptional set N compact or satisfying other negligibility conditions. As we note from the above, Theorems 1 and 2 are considerably sharper and more general than the Corollaries 2 stated above.

We now proceed to the proof of Theorem 1, which is based upon the following Lemma:

LEMMA. *Let Y be a Banach space, S_0 a bounded closed subset of Y, C a closed cone in Y generated by a closed bounded convex subset F of Y which does not contain 0. Then there exists an element s_0 of S_0 such that*

$$(s_0 + C) \cap S_0 = \{s_0\}.$$

The proof of the Lemma is given in §1 of Browder [4] and is based on an extension of the idea of the proof of the Bishop-Phelps Theorem [1].

PROOF OF THEOREM 1. Let $S=f(X)$, and suppose that $d_0=\text{dist}(y, S)>0$. We shall deduce a contradiction. For a given $\epsilon>0$, which we shall specify later in the proof, we may choose a point s in S such that

$$d = \|y - s\| \leqq (1 + \epsilon)d_0.$$

(If there exists a point s with $\|y-s\| = d_0$, we choose such an s in S and let $\epsilon = 0$.) By hypothesis, there exists $p<1$ such that for every x in $f^{-1}(B_r(y))$, there exists w in $D_x(f)$ such that if $\xi = \|y-f(x)\|$, then there exists w in $D_x(f)$ such that $\|\xi w - (y-f(x))\| \leqq p\xi$ with $0 \leqq p < 1$. We choose another constant q such that $0 \leqq p < q < 1$.

Let B be the closed ball of radius $r = qd_0$ about the point y in Y. Let K be the convex closure of the union of the point $\{s\}$ and the ball B. Then K is a closed bounded convex subset of Y, and u is any point of K, u may be written in the form

$$u = (1 - t)s + tz, \qquad (z \in B, t \in [0, 1]).$$

Let $S_0 = S \cap K$. Then S_0 is a closed bounded subset of Y. If u lies in S_0, then

$$d_0 \leqq \|u - y\| \leqq (1 - t)\|s - y\| + t\|z - y\| \leqq (1 - t)(1 + \epsilon)d_0 + tqd_0.$$

Hence

$$t \leqq \epsilon(\epsilon + (1 - q))^{-1}. \tag{1}$$

Let C be the closed cone with vertex at 0 in Y spanned by the closed bounded convex set $F = (B - s)$ which does not contain 0. If we apply the Lemma to the set S_0 and the cone C, it follows that there exists a point s_0 in S_0 such that $(s_0 + C) \cap S_0 = \{s_0\}$. Since s_0 lies in S_0, $s_0 = (1-t)s + tz$, with z in B and t in $[0, 1]$ satisfying the inequality (1) above. If y is an element of C, y can be written in the form

$$y = \xi(z_1 - s), \qquad (\xi \geqq 0, z_1 \in B).$$

Suppose that $y \neq 0$, and that $v = (s_0 + y)$ lies in S. Then:

$$\begin{aligned} v &= (1 - t)s + tz + \xi(z_1 - s) = (1 - t - \xi)s + tz + \xi z_1 \\ &= (1 - (t + \xi))s + (t + \xi)[t(t + \xi)^{-1}z + \xi(t + \xi)^{-1}z_1]. \end{aligned}$$

Suppose that

$$\xi \leqq (1 - q)(\epsilon + (1 - q))^{-1} = \delta, \qquad (\delta > 0).$$

Then $(t+\xi) \leqq t + \delta \leqq 1$, and v lies in K. Then we should have v in $S \cap K = S_0$, which contradicts the fact that $S_0 \cap (s_0 + C) = \{s_0\}$. Hence for any such v, $\xi > \delta$, so that we have $\|y\| = \xi\|z_1 - s\| > \delta(1-q)d_0 = \delta_1$. Thus,

$$(s_0 + C) \cap S \cap B_{\delta_1}(s_0) = \{s_0\}.$$

Hence, for any point x in X for which $s_0 = f(x)$, it follows that

$$D_x(f) \cap \operatorname{Int}(C) = \varnothing,$$

where $\mathrm{Int}(C)$ denotes the interior of the cone C in Y. For any such point x, we have

$$\|y - f(x)\| \leqq \|y - s\| + \|s - s_0\|,$$

with $(s - s_0) = t(s - z)$ in terms of the representation for s_0 considered above with z in B. Therefore,

$$\|y - f(x)\| \leqq (1 + \epsilon)d_0 + \epsilon(\epsilon + (1 - q))^{-1}(1 + \epsilon + q)d_0 = d_0 + \epsilon s d_0.$$

If the constant r of the hypothesis exceeds d_0, we may choose $\epsilon > 0$ so small that $d_0 + \epsilon s d_0 \leqq r$. If $r = d_0$, we choose $\epsilon = 0$, $s = s_0$, and x automatically lies in $f^{-1}(B_r(y))$. In both cases, we may choose ϵ sufficiently small so that x lies in $f^{-1}(B_r(y))$.

Finally we conclude the proof by deducing that $D_x(f) \cap \mathrm{Int}(C)$ is nonempty for small ϵ which contradicts our preceding argument. For the given point x, there exists w in $D_x(f)$ such that for $\xi = \|y - f(x)\| = \|y - s_0\|$, we have

$$\|\xi w - (y - s_0)\| \leqq p\xi,$$

i.e.

$$\|(s_0 + \xi w) - y\| \leqq p\|y - s_0\| \leqq p d_0 + \epsilon p s d_0.$$

We choose ϵ so small that $p + \epsilon p s < q$. Then $(s_0 + \xi w)$ lies in the interior of the ball B, i.e. ξw lies in the interior of $(B - s_0)$. Hence ξw lies in the interior of C, and so does w itself, i.e. $w \in D_x(f) \cap \mathrm{Int}(C)$.

This contradiction to the initial assumption that d_0 is positive establishes the validity of the theorem. q.e.d.

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