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BOOK REVIEWS

Convergence of probability measures by P. Billingsley. Translated by R. M. Dudley. Wiley Series in Probability and Mathematical Statistics, John Wiley, New York, 1968. xii+253 pp. $12.50.

The author's preface gives an outline: "This book is about weak-convergence methods in metric spaces, with applications sufficient to show their power and utility. The Introduction motivates the definitions and indicates how the theory will yield solutions to problems arising outside it. Chapter 1 sets out the basic general theorems, which are then specialized in Chapter 2 to the space $C[0, 1]$ of continuous functions on the unit interval and in Chapter 3 to the space $D[0, 1]$ of functions with discontinuities of the first kind. The results of the first three chapters are used in Chapter 4 to derive a variety of limit theorems for dependent sequences of random variables."

The book develops and expands on Donsker's 1951 and 1952 papers on the invariance principle and empirical distributions. The basic random variables remain real-valued although, of course, measures on $C[0, 1]$ and $D[0, 1]$ are vitally used. Within this framework, there are various possibilities for a different and apparently better treatment of the material.

More of the general theory of weak convergence of probabilities on separable metric spaces would be useful. Metrizability of the convergence is not brought up until late in the Appendix. The close relation of the Prokhorov metric and a metric for convergence in probability is (hence) not mentioned (see V. Strassen, Ann. Math. Statist. 36 (1965), 423–439; the reviewer, ibid. **39** (1968), 1563–1572). This relation would illuminate and organize such results as Theorems 4.1, 4.2 and 4.4 which give isolated, *ad hoc* connections between weak convergence of measures and nearness in probability.

In the middle of p. 16, it should be noted that $C^*(S)$ consists of signed measures which need only be finitely additive if S is not compact. On p. 239, where the author twice speaks of separable subsets having nonmeasurable cardinal, he means "discrete" rather than "separable."

Theorem 1.4 is Ulam's theorem that a Borel probability on a complete separable metric space is tight. Theorem 1 of Appendix 3 weakens completeness to topological completeness. After mentioning that probabilities on the rationals are tight, the author says it is an "open problem to characterize topologically those metric spaces S that support tight probability measures only." Given that S is separable, there is a criterion: that S be a measurable subset of its completion $\bar{S}$, for every Borel probability on $\bar{S}$. This may not be purely "topological" but it is perhaps the best solution one can expect. It suffices for S to be Borel or analytic in $\bar{S}$ (incidentally, the book does not mention analytic sets).

Theorem 6.1, that a uniformly tight set of probability measures is weakly relatively sequentially compact (highly useful, however it may sound), can be proved more elegantly using separability of the continuous functions on a compact metric space for the supremum norm. This theorem and its converse really go back to A. D. Alexandrov's work in the early 1940's, although Prokhorov's 1956 paper has been their springboard into wide knowledge and applications.

Pages 65–67 and 134–136 deal with separable stochastic processes. In each case a footnote says the topic "may be omitted." In fact, this notion of separability now seems entirely dispensable.

The book systematically uses the Skorokhod topology on $D[0, 1]$. I believe this topology has become entrenched after an excessive retreat from an oversight of Donsker's. For the distributions ν_n of empirical distribution functions, Donsker proved $\int F d\nu_n \rightarrow \int F d\nu$ for functionals F which are bounded and continuous a.e. (ν) for the supremum norm, except that F need not be measurable for ν_n. But for measurability it is *not* necessary to make F *continuous* for a much weaker topology. The Skorokhod topology is developed only for functions of one variable and does not make addition of functions continuous. I have shown that Donsker's original theorem extends to several variables (Illinois J. Math. **10** (1966), 109–126; **11** (1967), 449–453).

The final chapter presents mainly Billingsley's own research, much of it published here for the first time. In the Donsker limit theorems, independent random variables are replaced by strictly stationary sequences with suitable mixing properties (mixing amounts to a kind of approximate independence).

K. R. Parthasarathy's book *Probability measures on metric spaces* covers much of the same material, but differs in treating independent random variables taking values in locally compact Abelian groups and Hilbert spaces, proving relatively complete and final extensions of the classical theory of infinitely divisible laws.

The results Billingsley presents seem less final and polished than those on infinitely divisible measures. But this difference seems to result more from the greater difficulty of the field attacked than from any lesser power of technique. When one of the main researchers in a field takes the time and trouble to write an exposition, he makes a valuable contribution; certainly that is the case here.

R. M. DUDLEY

Geometric measure theory by Herbert Federer. Springer-Verlag, Berlin, Heidelberg, New York, 1969. 676 pp. $29.50.

Measure theory is perhaps the least honored of the several large mathematical disciplines which have been developed during the twentieth century.

A number of reasons may be given for this humble standing of the subject. In the first place, the French school of mathematicians, with its high prestige level and talent for persuasiveness, had relegated the subject to a relatively minor role and declared it to be a small branch of functional analysis, another discipline of rather low status, except perhaps in its applications to partial differential equations. A second reason is that the subject was largely regarded as a tool for probability theory and this, for a while, involved, for the most part, some of the pleasant but not especially deep or difficult aspects of measure theory. Thirdly, the representation of measure algebras consists primarily of one theorem, that a homogeneous nonatomic normal algebra is homeomorphic to the measure algebra of a product of circles, and this has an interesting but not especially difficult proof.

This is an improper assessment of measure theory. Probability theory has had an unexpectedly proliferous growth, giving more to other fields than it has taken. It is often indistinguishable from measure theory. It may suffice to note its connection with potential theory via brownian motion and more general processes, its elucidation of the behavior of orthonormal systems via martingales, and its generalization of differentiation theory.

A more serious slighting of measure theory by distinguished and influential people lay in their failure to emphasize that the deepest and perhaps most useful aspects of measure theory relate to those measures for which open sets have infinite measure. Geometric measure theory deals with measures of this sort. Recent developments, notable among which are the appearance of Federer's book, are bringing about a change in the position of measure theory described above. The subject deals with measures of lower dimensional sets imbedded in higher dimensional spaces, as well as with the evaluation and properties of integrals over appropriate lower dimensional domains. It accordingly includes such topics as the Gauss-Green theorem and Stokes' theorem, as well as matters related to the Plateau problem.