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FINITE DIMENSIONAL *H*-SPACES[1]

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1. Introduction. An *H-space* is a space X with base point e equipped with a continuous function

$$X \times X \xrightarrow{\mu} X$$

such that $\mu(x, e) = x = \mu(e, x)$ for all $x \in X$. This condition can also be formulated as follows. The *wedge* $X \vee X$ of X is defined by

$$X \vee X = (X \times e) \cup (e \times X) \subset X \times X,$$

and the *folding map* ∇ is given by

$$\nabla(x, e) = x, \qquad \nabla(e, x) = x, \qquad \nabla : X \vee X \to X.$$

The H-space condition is then that the following diagram commutes

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \uparrow{\scriptstyle i} & \nearrow{\scriptstyle \nabla} & \\ X \vee X & & \end{array}$$

where i is the inclusion map. This formulation has the advantage that we can now relax the condition and just require that the diagram be homotopy commutative; i.e., that the maps $\mu \circ i$ and are homotopic, $\mu \circ i \sim \nabla$.

When one is considering different H-space structures μ, μ' on an H-space X this distinction between the diagram commuting and homotopy commutating is important. But we will be concerned solely

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[3] The author felt no need to expand the talk to a survey of H-spaces because of the monograph of Stasheff, "H-spaces from the homotopy point of view," Springer, 1970. The only exceptions are §8, which notes some recent results not covered in the talk (nor in Stasheff's monograph), and a bibliography much more extensive than just references made in the paper.

with the question: Given X, does X support any H-space structure μ. Then the distinction is unimportant. For if X is nice enough to have the homotopy extension property and $\mu \circ i \sim \nabla$, then there exists a $\mu' : X \times X \to X$ with $\mu' \circ i = \nabla$. From this it also follows easily that if Y is an H-space and X has the homotopy type of Y, $X \sim Y$, then X is also an H-space.

An important example of H-spaces is the *space of loops* in a space. If $y_0 \in Y$, we define

$$\Omega(Y, y_0) = \{f: [0, 1] \to Y \mid f \text{ is continuous}, f(0) = f(1) = y_0\},$$

and topologize this set with the compact-open topology. An H-space structure $\mu : \Omega \times \Omega \to \Omega$ on Ω is given by

$$\begin{aligned} \mu(f, g)(t) &= f(2t), \qquad & 0 \leqq t \leqq 1/2, \\ &= g(2t - 1), \qquad & 1/2 \leqq t \leqq 1. \end{aligned}$$

Thus, in a vague sense, we have as many H-spaces as we have topological spaces. But loop spaces are usually infinite dimensional in a very real sense. For example, it is a classical result due to M. Morse that

$$H^{k(n-1)}(\Omega(S^n, y_0); \mathbf{Z}) \neq 0$$

for $k = 1, 2, 3, \cdots$.

So we will consider finite-dimensional H-spaces. As a matter of fact we consider *compact, simply connected manifolds* and ask which of these will support an H-space structure. For many of the important H-space questions we are ignoring (e.g. how many H-space structures a given H-space X has, does X have homotopy associative or homotopy commutative structures, etc.) the reader is referred to Stasheff's monograph.

We begin by recalling the classification of a very special class of H-spaces.

2. **Lie groups.** A *topological group* (X, μ) is an H-space such that μ is associative and such that left translations by elements a of X

$$L_a : X \to X$$
$$x \to \mu(a, x)$$

are homeomorphisms. Since the affirmative solution to Hilbert's fifth problem by Gleason and Montgomery-Zippin, we know that if X is a manifold which is a topological group, then X is actually a Lie group; i.e. it has a differentiable structure in which the group operations are analytic. Powerful analytic and algebraic tools have been used in the study of Lie groups and all Lie groups are known. All Lie groups can be built up from vector groups, tori and *simple* groups (those having no closed normal subgroups). All of the simple groups are known from the work of E. Cartan, Killing and Weyl.

The compact simply connected simple Lie groups are:

(i) The *spin groups* Spin(n), of dimension $n(n-1)/2$ (these are double coverings of the rotation groups $SO(n)$).

(ii) The *special unitary groups* SU(n), of dimension n^2-1.

(iii) The *symplectic groups* Sp(n), of dimension $2n^2+n$.

(iv) The five *exceptional groups*, G_2, F_4, E_6, E_7, E_8 of dimensions 14, 52, 78, 133, 248.

The existence of these exceptional groups is mysterious and one has the impression that he is looking at a small part of a jig-saw puzzle. It is a long-range hope that by studying H-spaces we will see more of the puzzle and that these groups will be a part of some reasonable pattern.

Incidentally, in classifying spaces which support a Lie group structure one need not worry about what kind of equivalence of spaces is used (e.g. homotopy equivalence, homeomorphism). It is a remarkable theorem due to Baum and Browder [8] that if two simple Lie groups G_1, G_2 are homotopy equivalent, then they are *isomorphic*. This result was extended by Scheerer [64] to all simply connected Lie groups.

3. **Equivalence of spaces.** For classifying H-spaces there is no result like the Baum-Browder result for Lie groups, and we must decide what is the most appropriate equivalence relation. If we insist on homeomorphism as an equivalence, we get too many H-spaces to make classification reasonable. This is pointed up sharply by the following theorem due to John Morgan (unpublished).

THEOREM. *Let M^n be a compact simply connected manifold with $n \geqq 6$. Then there exist infinitely many nonhomeomorphic manifolds $\{M_i\}$ each with the homotopy type of M if and only if*

$$H^{4k}(M^n; Q) \neq 0$$

for some $k>0$ with $4k<n$.

If we have an H-space with such cohomology, then all of the M_i will also be H-spaces. For example,

$$H^8(\mathrm{SU}(4); Q) \neq 0,$$

so these are infinitely many H-space manifolds all with the homotopy type of SU(4).

Thus we work with homotopy types—two manifolds are equivalent if they have the same homotopy type. This will eliminate the problem we had using homeomorphism as equivalence, for it is a recent result of Curjel-Douglas [26] that in any dimension n there are only finitely many homotopy types of spaces which will support an H-space structure.

In working toward a classification there are obviously two lines of attack.

A. Find conditions X must satisfy if it is to be an H-space.

B. Find new H-spaces.

We will be concerned with A in the next two sections.

4. Cohomology and homotopy of H-spaces. There is a classical result due to H. Hopf [38] about the rational cohomology of a (finite-dimensional) H-space X; namely,

$$H^*(X; Q) = \Lambda(x_1, \cdots, x_r),$$

that is, $H^*(X; Q)$ is an exterior algebra. Furthermore, the dimension of each generator x_i is odd.

(This contrasts sharply with the situation for infinite-dimensional H-spaces. For example, $\Omega(S^{2n+1}, y_0)$ has, as cohomology ring, a polynomial algebra on one generator of dimension $2n$.)

The number r of generators in

$$H^*(X; Q) = \Lambda(x_1, \cdots, x_r)$$

is called the *rank* of the H-space X. If X is a Lie group, this agrees with the usual definition of rank.

A. Borel [10] has calculated that Z_p cohomology of an H-space (p a prime)

$$H^*(X; Z_p) = \Lambda(x_1, \cdots, x_r) \oplus Z_p[y_1, \cdots, y_m]/(y_1^{p^{r_1}} \cdots y_m^{p^{r_m}}).$$

That is, it is a sum of an exterior algebra and a truncated polynomial algebra. If $p>2$ then the x_i have odd dimension and y_j have even dimension. If $p=2$ all generators have odd dimension.

These theorems make it easy to eliminate some candidates for H-spaces. For example, consider principal S^7 bundles over SU(3). By using the classifying bundle $S^7 \to S^{15} \to S^8$ and a simple Serre spectral sequence argument one sees that only the trivial bundle $S^7 \times \mathrm{SU}(3)$ has a possible cohomology ring to be an H-space. It is interesting to contrast this with a theorem of Curtis-Mislin [27] showing that all principal SU(3) bundles over S^7 are H-spaces.

W. Browder [17] proved that a finite-dimensional H-space X has $\pi_2(X)=0$. (For X a Lie group this had been proved by Hopf.) He also showed that X satisfies Poincaré Duality (without, of course, assuming, as we are, that our spaces are manifolds).

5. **Impossible rings.** A useful technique in eliminating candidates for H-spaces is based on the fact that *not all rings are realizable as cohomology rings of spaces.* For example, there does not exist a finite complex Y with torsion-free $\boldsymbol{Z}$-homology with the following cohomology ring (with $\boldsymbol{Z}$ coefficients)

H^2	H^4	H^6	H^8	H^{10}	H^{12}	H^{14}	H^{16}	H^{18}	H^{20}	H^{22}	H^{24}
0	0	0	$\boldsymbol{Z}$	0	$\boldsymbol{Z}$	0	$\boldsymbol{Z}$	0	$\boldsymbol{Z}$	0	$\boldsymbol{Z}$
generators:			x_1		x_2		x_3		x_4		x_5

The generators are required to satisfy

$$x_1^2 = x_3, \qquad x_2^2 = x_5, \qquad x_1x_2 = x_4.$$

The rest of the ring is arbitrary. This result has been proved independently by Hubbuck [41] and Douglas-Sigrist [30]. Its application to H-spaces is via the *Projective plane of an H-space*, a notion due to Stasheff [68]. We describe this briefly to show why the impossible ring given above was important.

Recall that the *Hopf construction* $H(f)$ on a map $f: A \times B \to C$ is the map of the join $A * B$ to the suspension ΣC given by the formula

$$H(f)(a, b, t) = (f(a, b), t).$$

The projective plane P_2X of an H-space X, $\mu: X \times X \to X$, is the mapping cone of $H(\mu): X * X \to \Sigma X$.

If X has dimension n, then P_2X has dimension $2n+2$. The term projective plane is justified somewhat by the following facts.

$$P_2(S^0) = \boldsymbol{R}P^2,$$
$$P_2(S^1) = \boldsymbol{C}P^2,$$
$$P_2(S^3) = HP^2 \text{ (quaternionic)},$$
$$P_2(S^7) = \text{Cayley projective plane.}$$

The cohomology of P_2X has been calculated by Browder and Thomas [20], and these results have been quite useful in eliminating certain candidates for H-spaces. Suppose some S^7 bundle over S^{11}, $S^7 \to X \to S^{11}$, were an H-space. Then $H^*(P_2X; \boldsymbol{Z})$ would be the impossible ring given at the beginning of this section. Thus no S^7 bundle over S^{11} is an H-space. Similarly one shows that no S^7 bundle over S^{15} is an H-space. These were the only cases of sphere bundles over spheres which had not already been decided by Frank Adams [2].

6. **Pre-Zabrodsky H-spaces.** We now come to part B, the construction of new H-spaces. Recall that we consider only compact simply connected manifolds. Before 1968 only S^7 and Lie groups (and, of course, products of these) were known. In 1968 the *Hilton-Roitberg Criminal* X [36], [37] appeared. The space X is a principal S^3 bundle over S^7 with the remarkable property that although X does not have the homotopy type of Sp(2), the spaces $X\times S^3$ and $\mathrm{Sp}(2)\times S^3$ are *diffeomorphic*. This shows X is an H-space since it is a retract of the Lie group $X\times S^3$, and it is easy to see that any retract of an H-space is an H-space. Unfortunately, the technique used here does not seem to work to produce more examples of H-spaces. But X was the first example (except S^7) which does not have the homotopy type of any Lie group.

Now there is a new method due to Zabrodsky [83] which produces scads of examples.

7. **Mixing homotopy types.** A map $f: X\to Y$ is a *rational equivalence* if

$$f^*: H^*(Y; Q) \to H^*(X; Q)$$

is an isomorphism. It is a p-equivalence if f^* is an isomorphism using Z_p as coefficients. If $\mathcal{P}$ is the set of all primes and $\mathcal{P}_1\subset\mathcal{P}$, a $\mathcal{P}_1$ *equivalence* means a p-equivalence for each $p\in\mathcal{P}_1$.

THEOREM (ZABRODSKY). *Let $f: X\to Y$ be a rational equivalence and let $\mathcal{P}_1\subset\mathcal{P}$. Then f may be factored as*

$$X \xrightarrow{f_1} X(\mathcal{P}_1) \xrightarrow{f_2} Y$$

where f_1 is a $\mathcal{P}_1$ (and rational) equivalence, f_2 is a $\mathcal{P}-\mathcal{P}_1$ (and rational) equivalence, and f_2 is a fibration. Moreover, if X and Y are H-spaces and f is an H-map, then $X(\mathcal{P}_1)$ is an H-space and f_1, f_2 are H-maps.

Instead of creating new H-spaces such as $X(\mathcal{P}_1)$, one may have a candidate at hand and want to establish that it is indeed an H-space. For that purpose the following corollary to Zabrodsky's theorem is useful.

COROLLARY [27]. *Given*

$$X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2,$$

f, g rational equivalences such that f is a $\mathcal{P}_1$-equivalence and g is a $\mathcal{P}_2$-equivalence with $\mathcal{P}_1\cup\mathcal{P}_2=\mathcal{P}$. Suppose X_0, X_2 are H-spaces having

homotopy types of finite CW complexes. Then X_1 is an H-space of the homotopy type of a finite CW complex.

Here is a simple example of how this corollary may be used.

THEOREM [27]. *All principal* SU(3) *bundles over S^7 are H-spaces.*

PROOF. Such bundles are classified up to bundle equivalence by homotopy classes of maps

$$f: S^7 \to B_{SU(3)}$$

and $\pi_7(B_{SU(3)}) = \pi_6(SU(3)) = \boldsymbol{Z}_6$ by a result due to R. Bott [11]. A generator $\alpha: S^7 \to B_{SU(3)}$ has pullback (=induced bundle)

$$SU(3) \to SU(4) \to S^7.$$

Let n denote a map $S^7 \to S^7$ of degree n. Let Y_n be the total space of the bundle induced by

$$S^7 \xrightarrow{n} S^7 \xrightarrow{\alpha} B_{SU(3)}.$$

We easily see that Y_n is homotopy equivalent to Y_{-n} (in $\boldsymbol{Z}_6$), so the only possible different homotopy types are

$$Y_0 = S^7 \times SU(3), \quad Y_1 = SU(4), \quad Y_2 \quad \text{and} \quad Y_3.$$

These are easily seen to be distinct homotopy types since $\pi_6(Y_n) = \boldsymbol{Z}_6/n\boldsymbol{Z}_6$. Consider the diagram

$$\begin{array}{ccccccc} Y_0 = & Y_6 & \xrightarrow{\phi} & Y_2 & \xrightarrow{\psi} & Y_1 & = SU(4) \\ & \downarrow & & \downarrow & & \downarrow & \\ & S^7 & \xrightarrow{3} & S^7 & \xrightarrow{2} & S^7 & \xrightarrow{\alpha} B_{SU(3)}. \end{array}$$

Then ϕ is a $\mathcal{P} - \{3\}$ equivalence and ψ is a $\mathcal{P} - \{2\}$ equivalence and both are rational equivalences. Since $Y_0 = S^7 \times SU(3)$ and $Y_1 = SU(4)$ are H-spaces, so is Y_2. Interchanging 2 and 3 we see that Y_3 is an H-space. q.e.d.

8. **Remarks.** This section is devoted to a report on a few recent developments in the theory of finite-dimensional H-spaces. References are not attempted because none of the results are in print at this time. Spaces here are to have the homotopy type of *finite CW complexes.*

(a) The classification of H-spaces of rank $\leqq 2$ (see §4) is now complete. In the simply connected case this takes us up through dimension 10. The homotopy types are:

$$S^3, (S^3)^2, S^7, \mathrm{SU}(3), (S^3)^3, S^3 \times S^7, \mathrm{Sp}(2) = E_\omega, E_{3\omega}, E_{4\omega}, E_{5\omega}.$$

Here ω is the map of S^7 into B_{S^3} inducing Sp(2) and $n\omega$ denotes a map $S^7 \to S^7$ of degree n following ω, $E_{n\omega}$ being the induced principal S^3 bundle over S^7. This work was done by Curtis-Mislin, Hilton-Roitberg and Zabrodsky, but only Zabrodsky succeeded in eliminating the candidates $E_{2\omega}$ and $E_{6\omega}$. Mislin has worked out the non-simply connected case, but his results in dimension 10 are not yet complete. Up through dimension 9 we have the following additions to the list above

$$S^1, PS^3, \mathrm{SO}(3), PS^7, RP^7, \mathrm{PSU}(3).$$

(b) Questions about associativity of the new 10-dimensional H-spaces are resolved as follows. $E_{5\omega}$ (which is the Hilton-Roitberg Criminal) has the homotopy type of a loop space; Stasheff [**69**]. $S^3 \times S^7$ and $E_{3\omega}$ admit no homotopy associative multiplication. $E_{4\omega}$ was more difficult, but D. Rector has recently shown it has no homotopy associative multiplication.

(c) A general result as to which of a special class of principal bundles over spheres are H-spaces has been obtained by Harrison and Stasheff, again using Zabrodsky's method of mixing homotopy types. Let $G \supset H$ be a pair of topological groups which are finite complexes. Assume that $G/H = S^n$ and that the element $\alpha \in \pi_{n-1}(H)$ classifying the principal bundle $H \to G \to S^n$ is of *finite order*. Write

$$\alpha = \alpha_2 + \alpha_3 + \alpha_5 + \cdots + \alpha_p + \cdots + \alpha_q$$

where subscripts represent distinct primes and α_p has pth power order. Let E_β be the bundle induced by $\beta \in \pi_{n-1}(H)$. Then suppose

$$\beta = \sum \epsilon_i \alpha_i$$

with $\epsilon_i = 0, \pm 1$. Then E_β is an H-space $\Leftrightarrow$

(i) n is odd and $\epsilon_2 \neq 0$, or

(ii) $n = 1, 3, 7$.

(d) A contribution to the knowledge of the $\mathbf{Z}_p$ cohomology of finite-dimensional topological groups X has recently been made by C. Wilkerson. By using Adams operations (see Hubbuck [**41**]), he generalized Serre's theorem for compact Lie groups, stating for which primes p, X is a p-equivalent to a product of spheres.

(e) Curjel has shown that a frequently used hypothesis about H-spaces is frequently superfluous. Writing

$$H^*(X; \mathbf{Q}) = \Lambda(x_1, \cdots, x_r),$$

he shows that by repeatedly changing the H-space structure on X, the generators $x_1, \cdots, x_r$ may all be made primitive, so that $H^*(X; \mathbf{Q})$ is primitively generated. The result extends to $H^*(X; \mathbf{Z})$ if it has no torsion.

Bibliography

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632. MR **25** #2614.

2. ———, *H-spaces with few cells*, Topology **1** (1962), 67–72. MR **26** #5574.

3. ———, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104. MR **25** #4530.

4. ———, *The sphere, considered as an H-space* mod p, Quart, J. Math. Oxford Ser. (2) **12** (1962), 52–60. MR **23** #A651.

5. S. Araki, I. M. James and E. Thomas, *Homotopyabelian Lie groups*, Bull. Amer. Soc. **66** (1960), 324-326. MR **22** #9973.

6. M. F. Atiyah, *Power operations in K-theory*, Quart. J. Math. Oxford Ser (2) **17** (1966), 165–193. MR **34** #2004.

7. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R. I., 1961, pp. 7–38. MR **25** #2617.

8. Paul F. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305–336. MR **32** #6490.

9. J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. **74** (1968), 1117–1122. MR **38** #5215.

10. A. Borel, *Sur la cohomologie des espaces fibrês principaux et des espaces homogènes de groups de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207. MR **14**, 490.

11. R. Bott, *A report on the unitary group*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R. I., 1961, pp. 1–6. MR **24** #A2393.

12. ———, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337. MR **22** #987.

13. W. Browder, *Homology operations and loop spaces*, Illinois J. Math. **4** (1960), 347–357. MR **22** #11395.

14. ———, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375. MR **27** #5260.

15. ———, *Homotopy commutative H-spaces*, Ann. of Math. (2) **75** (1962), 283–311. MR **27** #765.

16. ———, *The cohomology of covering spaces of H-spaces*, Bull. Amer. Math. Soc. **65** (1959), 140–141. MR **22** #1891.

17. ———, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51. MR **23** #A2201.

18. ———, *Loop spaces of H-spaces*, Bull. Amer. Math. Soc. **66** (1960), 316–319. MR **22** #8508.

19. W. Browder and Edwin Spanier, *H-spaces and duality*, Pacific J. Math. **12** (1962), 411–414. MR **26** #1891.

20. W. Browder and Emery Thomas, *On the projective plane of an H-space*, Illinois J. Math. **7** (1963), 492–502. MR **27** #1955.

21. Robert F. Brown, *H-manifolds have no nontrivial idempotents*, U.C.L.A., 1969 (preprint).

22. ———, *Divisible H-spaces*, U.C.L.A., 1969 (preprint).

23. Allan, Clark, *On π_3 of finite dimensional H-spaces*, Ann. of Math. (2) **78** (1963), 193–196. MR **27** #1956.

24. ———, *Hopf algebras over Dedekind domains and torsion in H-spaces*, Brown University, Providence, R. I. (preprint).

25. A. H. Copeland, Jr., *On H-spaces with two non-trivial homotopy groups*, Proc. Amer. Math. Soc. **8** (1957), 184–191. MR **18**, 754.

26. C. R. Curjel and P. R. Douglas, *On H-spaces of finite dimension*, (preprint).

27. Morton Curtis and Guido Mislin, *H-spaces which are bundles over S^7*, Rice University, Houston, Tex., 1969 (preprint).

28. A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305. MR **21** #331.

29. R. R. Douglas, *Homotopy-commutativity in H-spaces*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 263–283. MR **36** #3347.

30. R. R. Douglas and F. Sigrist, *Sphere bundles over spheres which are H-spaces*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **44** (1968), 502–505. MR **38** #5211.

31. R. R. Douglas, P. Hilton and F. Sigrist, *H-spaces*, Lecture Notes in Math., no. 92, Springer-Verlag, Berlin and New York, 1969, pp. 65–73.

32. R. R. Douglas and F. Sigrist, *Homotopy-associative H-spaces which are sphere bundles over spheres*, Comment. Math. Helv. **44** (1969), 308–309. MR **40** #2071.

33. ———, *Sphere bundles over spheres and H-spaces*, Topology **8** (1969), 115–118. MR **39** #6312.

34. E. Halpern, *Twisted polynomial hyperalgebras*, Mem. Amer. Math. Soc. No. 29 (1958). MR **21** #2982.

35. P. J. Hilton, *Note on a theorem of Stasheff*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 127–130. MR **25** #4536.

36. P. J. Hilton and J. Roitberg, *On principal S^3-bundles over spheres*, Ann. of Math. (2) **90** (1969) 91–107. MR **39** #7624.

37. ———, *Note on principal S^3-bundles*, Bull. Amer. Math. Soc. **74** (1968), 957–959. MR **37** #5889.

38. H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. (2) **42** (1941), 22–52. MR **3**, 61.

39. J. R. Hubbuck, *On homotopy commutative H-spaces*, (preprint).

40. ———, *On finitely generated cohomology Hopf algebras*, (preprint).

41. ———, *Generalized cohomology operations and H-spaces of low rank*, (preprint).

42. ———, *Some results in the theory of H-spaces*, Bull. Amer. Math. Soc. **74** (1968), 965–967. MR **37** #4805.

43. I. M. James, *Commutative products on spheres*, Proc. Cambridge Philos. Soc. **53** (1957), 63–68. MR **18**, 918.

44. ———, *The ten types of H-spaces*, 1959, (preprint).

45. ———, *On Lie groups and their homotopy groups*, Proc. Cambridge Philos. Soc. **55** (1959), 244–247. MR **23** #A1371.

46. ———, *On homotopy-commutativity*, Topology **6** (1967), 405–410. MR **35** #3662.

47. ———, *On spaces with a multiplication*, Pacific J. Math. **7** (1957), 1083–1100. MR **19**, 876.

48. ———, *On H-spaces and their homotopy groups*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 161–179. MR **24** #A2966.

49. ———, *Multiplication on spheres*. I, Proc. Amer. Math. Soc. **8** (1957), 192–196. MR **18**, 752.

50. ———, *Multiplication on spheres*. II, Trans. Amer. Math. Soc. **84** (1957), 545–558. MR **19**, 875.

51. ———, *Products on spheres*, Mathematika **6** (1959), 1–13. MR **26** #5576.

52. ———, *On sphere-bundles*. I, Bull. Amer. Math. Soc. **75** (1969), 617–621. MR **39** #3517.

53. I. M. James and E. Thomas, *Homotopy-abelian topological groups*, Topology **1** (1962), 237–240. MR **26** #6970.

54. ———, *Which Lie groups are homotopy-abelian?* Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 737–740. MR **21** #5197.

55. ———, *On homotopy-commutativity*, Ann. of Math. (2) **76** (1962), 9–17. MR **25** #2603a.

56. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres*. II, Proc. London Math. Soc. (3) **5** (1955), 148–166. MR **16**, 948.

57. ———, *The homotopy theory of sphere bundles over spheres*. I, Proc. London Math. (3) **4** (1954), 196–218. MR **15**, 892.

58. M. A. Kervaire, *On the Pontryagin classes of certain* SO(n)*-bundles over manifolds*, Amer. J. Math. **80** (1958), 632–638. MR **21** #1592.

59. R. J. Milgram, *The bar construction and abelian H-spaces*, Illinois J. Math. **11** (1967), 242–250. MR **34** #8404.

60. J. W. Milnor, and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR **30** #4259.

61. John Morgan, *Stable tangential homotopy equivalences*, Ph.D. Thesis, Rice University, Houston, Tex., 1969.

62. F. P. Peterson, *A note on H-spaces*, Bol. Soc. Mat. Mexicana (2) **4** (1959), 30–31. MR **22** #11393.

63. H. Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. **58** (1952), 2–37. MR **13**, 533.

64. Hans Scheerer, *Homotopieäquivalente kompakte Liesche Gruppen*, Topology **7** (1968), 227–232. MR **37** #4833.

65. J.-P. Serre, *Quelques calculs de groupes d'homotopie*, C. R. Acad. Sci. Paris **236** (1953), 2475–2477. MR **14**, 1110.

66. Larry Smith, *On the type of an associative H-space of rank two*, Tôhoku Math. J. (2) **20** (1968), 511–515. MR **39** #4844.

67. E. H. Spanier and J. H. C. Whitehead, *On fibre spaces in which the fibre is contractible*, Comment, Math. Helv. **29** (1955), 1–8. MR **16**, 610.

68. J. D. Stasheff, *On homotopy Abelian H-spaces*, Proc. Cambridge Philos. Soc. **57** (1961), 734–745. MR **25** #4534.

69. ———, *Manifolds of the homotopy type of (non-Lie) groups*, Bull. Amer. Math. Soc. **75** (1969), 998–1000. MR **39** #7623.

70. ———, *Homotopy associativity of H-spaces*. I, II, Trans. Amer. Math. Soc. **108** (1963), 275–312. MR **28** #1623.

71. M. Sugawara, *A condition that a space is an H-space*, Math. J. Okayama Univ. **6** (1957), 109–129. MR **19**, 160.

72. ———, *A condition that a space is group-like*, Math. J. Okayama Univ. **7** (1957), 123–149. MR **20** #3546.

73. ———, *On the homotopy-commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **33** (1960/61), 257–269. MR **22** #11394.

74. ———, *Some remarks on homotopy equivalences and H-spaces*, Math. J. Okayama Univ. **8** (1958), 125–131. MR **21** #2981.

75. Emery Thomas, *Steenrod squares and H-spaces*, Ann. of Math. (2) **77** (1963), 306–317. MR **26** #3057.

76. ———, *Steenrod squares and H-spaces*. II, Ann. of Math. (2) **81** (1965), 473–495. MR **31** #1671.

77. ———, *The Steenrod squares in the mod two cohomology algebra of an H-space*, Colloque de Topologie, (Brussels, 1964), Librairie Universitaire, Louvain, 1966, pp. 113–117. MR **36** #3342.

78. H. Toda, *Generalized Whitehead products and homotopy groups of spheres*, J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3** (1952), 43–82. MR **15**, 732.

79. ———, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N. J., 1962. MR **26** #777.

80. G. W. Whitehead, *Note on a theorem of Sugawara*, Bol. Soc. Mat. Mexicana (2) **4** (1959), 33–41. MR **23** #A648.

81. ———, *On mappings into group-like spaces*, Comment. Math. Helv. **28** (1954), 320–328. MR **16**, 505.

82. A. Zabrodsky, *Implications in the cohomology of H-spaces*, (mimeograph).

83. ———, *Homotopy associativity and finite* CW *complexes*, University of Illinois, Chicago Circle, Urbana, Ill. (mimeograph).

84. ———, *The classification of simply connected H-spaces with three cells*, (to appear).

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